Landau singularities and singularities of holonomic integrals of the Ising class

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Abstract.
We consider families of multiple and simple integrals of the "Ising class" and the linear ordinary differential equations with polynomial coefficients they are solutions of. We compare the full set of singularities given by the roots of the head polynomial of these linear ODE's and the subset of singularities occurring in the integrals, with the singularities obtained from the Landau conditions. For these Ising class integrals, we show that the Landau conditions can be worked out, either to give the singularities of the corresponding linear differential equation or the singularities occurring in the integral. The singular behavior of these integrals is obtained in the self-dual variable $w = s/2/(1+s^2)$, with $s = \sinh(2K)$, where $K = J/kT$ is the usual Ising model coupling constant. Switching to the variable s, we show that the singularities of the analytic continuation of series expansions of these integrals actually break the Kramers-Wannier duality. We revisit the singular behavior (J. Phys. A **38** (2005) 9439-9474) of the third contribution to the magnetic susceptibility of Ising model $\chi^{(3)}$ at the points $1+3w+4w^2 = 0$ and show that $\chi^{(3)}(s)$ is not singular at the corresponding points inside the unit circle $|s| = 1$, while its analytical continuation in the variable s is actually singular at the corresponding points $2+s+s^2 = 0$ oustside the unit circle ($|s| > 1$).

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1. Introduction

Understanding the magnetic susceptibility χ of the Ising model as a function of the temperature remains an outstanding problem in lattice statistical physics. One of the problems raised in this model is the possibility of occurrence for χ of a natural

boundary in the complex plane of the variable $s = \sinh(2K)$, where $K = J/kT$ is the conventional Ising model coupling constant. There is a strong argument on this point, since it has been shown by Nickel [1, 2] that the singularities of each $n-$particle contribution $\chi^{(n)}$ to the susceptibility are given by

$$s + \frac{1}{s} \, = \, u^k + u^{-k} \, + u^m + u^{-m}, \tag{1}$$
$$\text{with:} \qquad u^n \, = \, 1, \qquad -n \le k, \quad m \le n$$

which are actually lying on the unit circle $|s| = 1$. In the following , we will call these singularities "nickellian singularities". These individual $\chi^{(n)}$ are $n-$particle contributions to the infinite sum [3]

$$\chi \, = \, \sum \chi^{(n)} \tag{2}$$

where the singularities (1) accumulate and (in absence of cancellation) densify the unit circle, possibly creating a natural boundary for χ. The above sum is restricted to n even (respectively odd) for the low (respectively high) temperature.

Among these $n-$particle contributions, $\chi^{(1)}$ and $\chi^{(2)}$ were known long ago, and the following two, $\chi^{(3)}$ and $\chi^{(4)}$, have been shown recently [4, 5, 6], to satisfy Fuchsian linear differential equations of, respectively, order seven and order ten.

Our study is based on a *holonomic* approach which uses the fact that, the integrand being an *algebraic function* of all the variables, the multiple integral under consideration, can be shown to be *holonomic* [7], i.e. it satisfies finite order linear differential equations, with polynomial coefficients, in the remaining variable for which no integration has been performed. The analysis, then, focuses on the corresponding linear differential operators. This holonomic approach gives the full set of singularities as roots of the head polynomial in front of the highest derivative of the differential equation (excluding the "apparent" polynomial [4, 5, 6]).

The linear ODE corresponding to $\chi^{(4)}$ has [6], besides the known singularities $s^2 = 1$, and $s^2 = -1$, only the singularities (1) predicted by Nickel [1, 2], while the linear ODE corresponding to $\chi^{(3)}$ has shown a pair of singularities $1 + 3w + 4w^2 = 0$ (where $w = \, s/2/(1+s^2)$) that are *not* on the unit circle $|s| = 1$. The condition $|s| = 1$ is equivalent, for the variable w, to be real and such that its absolute value is greater than $1/4$. In this paper, by w is "in $\mathcal{W}_c$", we will mean $w \in] - \infty, -1/4] \cup [1/4, \infty[$.

Therefore, the following question is in order. Should we expect in the linear ODE's corresponding to the successive n-particle contribution $\chi^{(n)}$, the occurrence of singularities that are of similar nature as $1 \, + 3w \, + 4w^2 \, = 0$, i.e. not lying in $\mathcal{W}_c$, or only the occurrence of the "nickellian singularities" given by (1)?

These expected singularities of the linear ODE governing the $\chi^{(n)}$ are not necessarily singularities of the particular solution $\chi^{(n)}$. Do singularities with $|s| > 1$ occur in the $\chi^{(n)}$, and accumulate, the question of a natural boundary for χ becomes of the most importance.

In previous works, we have introduced a "connection matrix method" [8] to determine if a singularity of the linear ODE is actually a singularity occurring in the multiple integral. For the three-particle contribution $\chi^{(3)}$, we have shown [8] that the two quadratic roots of $1 + 3w + 4w^2 \, = \, 0$, singularities of the linear ODE, are actually *absent in the multiple integral defining* $\chi^{(3)}$ written as a function of the self-dual variable w‡.

‡ Be aware that some subtlety can be encountered when the variable s is used, see Section (3.3) and Section (4.5).

In our holonomic approach, the singularities of the multiple integrals are, thus, obtained in two steps: find the linear differential equation, then use our "connection method" [8] in order to analyze the singularities of the solutions of this ODE and, especially, the singularities of that particular solution corresponding to the multiple integral. These computations may be rather cumbersome, depending on the order of the linear ODE, on the number of singularities, and their distribution in the complex plane of the variable.

In particle physics, in the study of analytical properties of the S-matrix [9], or in Feynman diagram calculations, the amplitudes of the Green functions are defined through an integral representation and depend on external kinematical invariants (the Mandelstam variables), the quadri-momentum in the loops being the integration variables. The singularities of these integrals are associated with physical thresholds in each of the physical regions in which they describe the process under consideration. The question of forecasting the location of the singularities of functions defined through integral representations is, thus, a general and interesting problem in mathematical physics [10].

An *analytical approach* [9] of this issue provides, by imposing conditions on the integrand, singled-out sets of complex points, as *possible singularities* of the *integral*. These conditions on the integrand are called *Landau conditions* [9, 11, 12]. The singularities obtained this way are called *Landau singularities*. Note that, the Landau singularities were introduced to be those of the *integral*, and that, the Landau conditions are only *necessary conditions* for the existence of such singularities [9, 12, 13]. In fact, when this analytical S-matrix approach was flourishing in the litterature, it was not yet known [7] that the integrals of (physical) interest *were actually holonomic*, that is, solutions of finite order linear differential equations with polynomial coefficients. The fact that the Landau singularities could have a better overlap with the singularities of the linear ODE's, rather than with the singularities of the integral of interest, was a question nobody could imagine at that time.

In many domains of mathematical physics [14], the computation of multiple integrals like the evaluation of Feynman diagrams [10], or various correlation functions in quantum field theory and statistical mechanics [15, 16] is an important problem. The n-particle contribution to the magnetic susceptibility of the Ising model have integral representations. They are given by $(n-1)$-dimensional integrals [2, 17, 18] that read (omitting the prefactor)

$$\tilde{\chi}^{(n)} \;=\; \frac{1}{n!}\cdot\Big(\prod_{j=1}^{n-1}\int_0^{2\pi}\frac{d\phi_j}{2\pi}\Big)\Big(\prod_{j=1}^{n} y_j\Big)\cdot R^{(n)}\cdot\;\Big(G^{(n)}\Big)^2, \qquad (3)$$

where

$$G^{(n)} \;=\; \Big(\prod_{j=1}^{n} x_j\Big)^{(n-1)/2}\cdot\prod_{1\le i<j\le n}\frac{2\sin\left((\phi_i-\phi_j)/2\right)}{1-x_ix_j}, \qquad (4)$$

and

$$R^{(n)} \;=\; \frac{1+\prod_{i=1}^{n} x_i}{1-\prod_{i=1}^{n} x_i}, \qquad (5)$$

with¶

$$x_i \;=\; \frac{2w}{1-2w\cos(\phi_i)\,+\sqrt{\left(1-2w\cos(\phi_i)\right)^2-4w^2}}, \qquad (6)$$

¶ Variables x_i and y_i are the variables noted $\tilde{x}_i$, and $\tilde{y}_i$, in previous papers [4, 5, 6].

$$y_i = \frac{2w}{\sqrt{(1 - 2w\cos(\phi_i))^2 - 4w^2}}, \qquad \sum_{j=1}^{n} \phi_j = 0 \tag{7}$$

valid for small w and, elsewhere, by analytical continuation. Recall that small w corresponds to small values of s, *as well as* large values of s.

If successful, the Landau conditions will give the singularities of these integrals bypassing the linear ODE search and the singularity analyzis from our connection method [8], possibly providing a deeper insight on the natural boundary problem of the susceptibility.

Unfortunatly, the multiple integrals (3) are too involved for the Landau conditions to be worked out. We will consider, instead, and as a preliminary project, two kinds of functions in the form of integral representations (one-dimensional and multidimensional) which belong to the Ising class♯. The integrands of this class of integrals have a trigonometric-hyperbolic form and are simple algebraic functions in the cosinus of the angular integration variables, and are also algebraic in the remaining w variable. These integrals are thus holonomic: they must satisfy finite order linear ODE's with polynomial coefficients in the remaining variable w. The route we take for obtaining the linear ODE's of the two sets of integrals will be through series expansions [4, 5, 6].

One example corresponds to a multiple integral, but yields quite simple sets of singularities, whereas the second example corresponds to simple integrals, but yields quite non trivial sets of singularities. These two "Ising class" examples have been chosen to provide an "equivalent level of complexity" but complementary results on our singularity problem. For each example, we compare the singularities of the linear ODE, the singularities of the integral, and the singularities given by the Landau conditions.

The paper is organized as follows. In Section (2), we recall the Landau equations method [9, 19]. Section (3) presents the first "toy model" for which we can obtain for *any value* n of this n-multiple integral, the corresponding linear ODE, remarkably of second order, and the corresponding solutions. Section (4) introduces a second "toy model" of simple integrals, gives the corresponding linear ODE's with their singularities, presents the Landau singularities, and compares these two sets with the actual singularities of the integral. In both sections, we consider some subtleties that appear when we switch from the self-dual variable w to the variable s, with respect to the analytic continuations of each. We finally comment the calculations on Landau conditions, sketch the forthcoming "Ising class" integral calculations we will study, and conclude.

2. Landau conditions

For the paper to be self contained, we recall, in this section, general features about the Landau singularities. Let us, for instance, consider the following integral:

$$F(w) = \int_D d\phi_1 \cdots d\phi_n \cdot f(w, \phi_1, \cdots, \phi_n) \tag{8}$$

where $f(w, \phi_1, \cdots, \phi_n)$ denotes an algebraic function of w and of trigonometric functions of the angles ϕ_i. Denote by $B_j\,(w, \phi)$, $j = 1, 2, \cdots d$, the set representing the boundary of the integration domain, and by $S_j\,(w, \phi)$, $j = 1, 2, \cdots s$, the set of varieties,

♯ The terminology *integral of the Ising class* has also been used by Bailey, Borwein and Crandall in [14].

locus of the singularities of the integrand (ϕ denotes $\phi_1, \phi_2, \cdots, \phi_n$). Landau equations amount to finding the necessary conditions for the singularities to occur when the hypercontour is pinched between the surfaces of singularities (pinch singularity) or meets a boundary variety (end-point singularity). For this to happen, the parameters α_i and β_i should exist, not all equal to zero, and such that at the point (w, ϕ)

$$\begin{aligned}
&\alpha_j \cdot B_j(w, \phi) = 0, \qquad j = 1, \cdots, d \\
&\beta_j \cdot S_j(w, \phi) = 0, \qquad j = 1, \cdots, s \\
&\frac{\partial}{\partial \phi_j}\Big(\sum_i \alpha_i \cdot B_i(w, \phi) + \sum_i \beta_i \cdot S_i(w, \phi)\Big) = 0, \qquad j = 1, \cdots, n
\end{aligned} \tag{9}$$

Solving this set of equations will give the singularities w that are "candidates" to be those of the integral (8). Note that the easy resolution of these equations will depend on the number of integration variables and the number of singularity varieties.

Consider, for instance, a one-dimensional integral in the form:

$$I_1 \; = \; \int_0^{2\,\pi} \frac{d\theta}{D(\theta, \, w)} \tag{10}$$

where $D(\theta, \, w)$ represents the variety of singularities. From (9), the Landau conditions read¶:

$$\begin{aligned}
&\alpha \cdot D(\theta, \, w) \; = \; 0, \qquad\qquad \beta \cdot \theta \; = \; 0, \\
&\frac{\partial}{\partial \theta}\Big(\alpha \cdot D(\theta, \, w) \; + \; \beta \cdot \theta\Big) = \; 0
\end{aligned} \tag{11}$$

Knowing that α and β should not be both equal to zero, it is easy to see that the set of equations (11) gives two kinds of solutions, namely:

$$D(\theta, \, w) \; = \; 0, \qquad\qquad \theta \; = \; 0 \quad (\mathrm{mod}\ 2\pi) \tag{12}$$

$$\text{or:} \quad D(\theta, \, w) \; = \; 0, \qquad\qquad \frac{\partial}{\partial \theta} D(\theta, \, w) \; = \; 0 \tag{13}$$

The first solution corresponds to *endpoint singularities* while the second one corresponds to *pinched singularities*. Note that mixed situations with endpoint together with pinched singularities, can occur for multidimensional integrals.

3. A multiple integral: the y^2-product

Recall that a multidimensional integral of the product of the y_i quantities has been recently introduced by Nickel [13] to suggest that the singularities, like $1 + 3w + 4w^2 = 0$, appearing in [4] for the $\chi^{(3)}$ linear differential equation are *pinch singularities*.

In this section, for elegance, and less prohibitive calculations purposes, we consider, instead, a multidimensional integral of the product of y_i^2's. The results are similar to the product of the y_i's, as far as the singularities location is concerned (see Appendix A for the linear ODE bearing on the product of y_i with $n = 3$).

Our multidimensional integral thus reads:

$$Y^{(n)}(w) \; = \; \int_0^{2\pi} \frac{d\phi_1}{2\pi} \int_0^{2\pi} \frac{d\phi_2}{2\pi} \, ... \int_0^{2\pi} \frac{d\phi_{n-1}}{2\pi} \left(\prod_{i=1}^{n} y_i^2\right) \tag{14}$$

¶ Strictly speaking, one has to add $\beta(\theta - 2\pi) = 0$, but this equation does not change the discussion. We are considering 2π periodic integrands.

with

$$\sum_{k=1}^{n} \phi_k \; = \; 0 \tag{15}$$

To find the linear differential equations for these integrals, we make use of the Fourier expansion of y_i^2 which reads (dropping the indices)

$$\begin{aligned} \frac{y^2}{4w^2} \; &= \; D(0) \; + 2 \sum_{k=1}^{\infty} D(k) \cdot \cos(k\phi) \\ &= \; \sum_{k=-\infty}^{\infty} D(k) Z^k \; = \; \sum_{k=-\infty}^{\infty} D(k) \cdot Z^{-k} \end{aligned} \tag{16}$$

with $Z = \exp(i\,\phi)$ and

$$D(k) \; = \; D(-k) = \; w^{|k|} \cdot d(|k|) \tag{17}$$

where $d(k)$ is a *non terminating* hypergeometric series given by

$$\begin{aligned} d(k) \; &= \; (1+k) \times \\ & {}_4F_3\Big(\frac{3}{4} + \frac{k}{2}, 1 + \frac{k}{2}, \frac{5}{4} + \frac{k}{2}, \frac{3}{2} \; + \frac{k}{2}; \frac{3}{2}, 1+k, \frac{3}{2} + k; 16\,w^2\Big) \end{aligned} \tag{18}$$

The Fourier expansion (16), together with the definitions (17), (18), allow to write the expansion of the integral $Y^{(n)}(w)$, fully integrated over the angles, as (see Appendix B for details)

$$Y^{(n)}(w) \; = \; (4w^2)^n \cdot \sum_{k=0}^{\infty} C_k \cdot w^{n\,k} \cdot d(k)^n \tag{19}$$

where $C_0 \, = \, 1$ and $C_k = \, 2$ for $k \, \geq \, 1$.

3.1. Linear ODE's and their solutions

Series generation using (19) can be carried out without any difficulty. We have computed the linear differential equations satisfied by the $Y^{(n)}$'s up to $n = 10$. These Fuchsian linear ODE's have many remarkable properties. They are of order *two*, *independently* of the dimension n of the integral. All these Fuchsian linear ODE's have the singularities $w = 0$, $w = 1/4$, $w = -1/4$ and ∞, with respectively the critical exponents $(2n, n-(-1)^n)$, $(-(n-2)/2, -(n+1)/2)$, $(-(n-2)/2, -(n+(-1)^n)/2)$ and $(-(n-1)/2, -(n+1)/2)$. The singularity $w = 0$ is an apparent one. These linear ODE's have other regular singularities with critical exponents -1 or 0.

Let us give, as an example, the linear ODE for $n = 3$ ($F(w)$ denotes $Y^{(3)}(w)$)

$$\sum_{m=0}^{2} a_m(w) \cdot \frac{d^m}{dw^m}\, F(w) \;\; = \;\; 0 \tag{20}$$

where

$$\begin{aligned} a_2(w) \; &= \; w^2 \cdot (1-w)\; (1+3\,w) \left(1 - 16\,w^2\right)^2 \\ & \qquad \times (1-5\,w) \left(1 + 3\,w + 4\,w^2\right) \cdot P_2(w), \\ a_1(w) \; &= \; \left(1 - 16\,w^2\right)\; w \cdot P_1(w), \qquad\qquad a_0(w) \; = \; 6\,P_0(w) \end{aligned}$$

the polynomials being:

$$\begin{aligned} P_2 \,&=\, 1 - 2\,w - 24\,w^2 - 5\,w^3 + 19\,w^4 + 98\,w^5 - 574\,w^6 - 2835\,w^7 \\ &\quad + 2330\,w^8 + 392\,w^9, \\ P_1 \,&=\, -9 + 16\,w + 410\,w^2 - 8\,w^3 - 5164\,w^4 - 7196\,w^5 + 12896\,w^6 \\ &\quad + 65824\,w^7 - 192637\,w^8 - 1302860\,w^9 + 327886\,w^{10} \\ &\quad + 6666128\,w^{11} + 3469546\,w^{12} - 6576560\,w^{13} \\ &\quad - 2874016\,w^{14} + 1967104\,w^{15} + 752640\,w^{16}, \\ P_0 \,&=\, 4 - 6\,w - 224\,w^2 + 29\,w^3 + 3808\,w^4 + 3272\,w^5 - 21680\,w^6 \\ &\quad - 71496\,w^7 + 87964\,w^8 + 1146670\,w^9 - 856344\,w^{10} \\ &\quad - 9773745\,w^{11} + 2595240\,w^{12} + 30066076\,w^{13} + 1626384\,w^{14} \\ &\quad - 26569152\,w^{15} - 8783616\,w^{16} - 1204224\,w^{17} + 2007040\,w^{18}. \end{aligned}$$

The roots of polynomial $P_2(w)$ are *apparent singularities*. The linear ODE for $n = 4$ is given in Appendix C.

This second-order linear differential equation can be solved exactly, and has the general solution (c_1 and c_2 are arbitrary constants):

$$\begin{aligned} F(w) \,&=\, c_1 \cdot F_1(w) \; + c_2 \cdot F_2(w), \qquad (21) \\ F_1(w) \,&= \\ &\frac{\left(3 - 59\,w^2 - 82\,w^3 + 386\,w^4 + 8\,w^5 + 64\,w^6\right) \cdot w^4}{3\,\left(1-4\,w\right)^2 (1-w)\,(1+3\,w)\,(1+4\,w)\,(1-5\,w)\,(1+3\,w+4\,w^2)}, \\ F_2(w) \,&=\, \frac{\left(1 + 2\,w - 2\,w^2\right) \cdot w^4}{(1-5\,w)\,(1+3\,w+4\,w^2)\,\sqrt{1-16\,w^2}} \end{aligned}$$

Each component $F_i(w)$ is a solution of an order-one linear differential equation. All the singularities are poles, except $w = \pm 1/4$ which are branch points for the second component $F_2(w)$. For arbitrary constants of integration, the general solution $F(w)$ has the same singular behavior. To obtain the particular solution corresponding to the integral (14) for $n = 3$, i.e. $F(w) = Y^{(3)}(w)$, the constants of integration have to be fixed to $c_1 = 12$, $c_2 = -c_1$. Note that, when these constants of integration are fixed at these values, the solution $F(w) = Y^{(3)}(w)$ is *no longer singular* at $w = 1/5$, and at the quadratic roots w_0 of $1 + 3w + 4w^2 = 0$, as can be easily checked:

$$\begin{aligned} Y^{(3)}(w) &= \frac{37}{1100}, \qquad \text{when} \qquad w \rightarrow 1/5 \qquad (22) \\ Y^{(3)}(w) &= \frac{4099}{90112} \pm i\,\frac{3881\sqrt{7}}{90112}, \qquad \text{when} \qquad w \rightarrow w_0 \end{aligned}$$

However, these polynomials $1 - 5\,w$ and $1 + 3\,w + 4\,w^2$ still remain in the expression of the solution. Note that the corresponding singularities $1 - 5\,w = 0$ and $1 + 3\,w + 4\,w^2 = 0$ *are not* in $\mathcal{W}_c$, while the others in (21) are.

As a second example, consider, for instance, the case $n = 5$. The singularities of the linear ODE, besides $1 - 16w^2 = 0$, correspond to the roots of the two polynomials:

$$\begin{aligned} Q_1^{(5)} &= \left(1 - 3\,w + w^2\right)\left(1 + 5\,w + 5\,w^2\right), \qquad (23) \\ Q_2^{(5)} &= \left(1 + 5\,w + 13\,w^2\right)\left(1 - 7\,w + 5\,w^2 - 4\,w^3\right) \\ &\quad \left(1 - 8\,w + 20\,w^2 - 17\,w^3\right)\left(1 + 8\,w + 20\,w^2 + 15\,w^3 + 4\,w^4\right). \end{aligned}$$

One can check that the set of values of w such that $1-16w^2 = 0$, and the singularities of the set $Q_1^{(5)} = 0$, are indeed singularities of the multiple integral $Y^{(5)}$ (see below and Appendix C), while those of the set $Q_2^{(5)} = 0$ are not. Here also, the singularities cannot be "simplified" due to the presence of the radical $\sqrt{1-16w^2}$. One can also check that the singularities corresponding to $Q_1^{(5)} = 0$ are in $\mathcal{W}_c$, while those of the set $Q_2^{(5)} = 0$ are not. These remarkable facts hold for all the other $Y^{(n)}$'s.

With the differential equations up to $n = 10$, we find that the singularities are roots of the following polynomials given by Nickel [13]

$$\cos\left(k\phi^+\right) - \cos\left((n-k)\phi^-\right) \;=\; 0, \tag{24}$$

$$\cos\left(\phi^+\right) \;=\; 1 + \frac{1}{2w}, \qquad \cos\left(\phi^-\right) \;=\; -1 + \frac{1}{2w} \tag{25}$$

which can be written as ¶:

$$T_k\Big(\frac{1}{2w}+1\Big) - T_{n-k}\Big(\frac{1}{2w}-1\Big) \;=\; 0, \qquad k \;=\; 0,\, 1,\, \cdots,\, n \tag{26}$$

where $T_k(w)$ are the Chebyshev polynomials of first kind. Note that, in this model, the singularity $1+4w = 0$ is not given by this formula for n odd. From these singularities, *only those* corresponding to $k = 0$, and $k = n$, *are singularities* of the $Y^{(n)}$'s, i.e.

$$\Big(1 - T_n\Big(\frac{1}{2w}-1\Big)\Big) \cdot \Big(1 - T_n\Big(\frac{1}{2w}+1\Big)\Big) \;=\; 0. \tag{27}$$

We are, now, in position to write the generic structure of the solutions $Y^{(n)}(w)$. For n odd, the solutions $Y^{(n)}(w)$ have the form:

$$Y^{(n)}(w) \;= \tag{28}$$
$$\frac{w^{n+1}}{(1-16\,w^2)^{n/2-1}\cdot Q_2^{(n)}} \cdot \Big(\frac{c_1^{(n)}\cdot R_1^{(n)}}{\sqrt{1-16\,w^2}\,(1-4\,w)\cdot Q_1^{(n)}} + c_2^{(n)}\cdot R_2^{(n)}\Big)$$

For n even, they read:

$$Y^{(n)}(w) \;=\; \frac{w^{n-1}}{\Big((1-16\,w^2)^{n/2-1}\Big)\cdot Q_2^{(n)}(w)} \;\times$$
$$\Bigg(\frac{c_1^{(n)}\cdot R^{(n)}(w)}{(1+4\,w)^{3/2}\cdot Q_1^{(n)}(w)} + \frac{c_2^{(n)}\cdot R^{(n)}(-w)}{(1-4\,w)^{3/2}\cdot Q_1^{(n)}(-w)}\Bigg)$$

The coefficients $c_1^{(n)}$ and $c_2^{(n)}$ are integration constants. The polynomials $R^{(n)}$, $R_1^{(n)}$ and $R_2^{(n)}$, in these expressions, have rational coefficients and are such that the constant coefficient in w is equal to 1. The polynomials $Q_1^{(n)}$ and $Q_2^{(n)}$ are constructed, respectively, from the roots of (27) and from the roots of the cumulative product (for $k = 1, \cdots, n-1$) of (26). The set of roots $Q_1^{(n)} = 0$ (resp. $Q_2^{(n)} = 0$) are (resp. are not) in $\mathcal{W}_c$.

The general solutions above give the particular solution (14) for $c_2^{(n)} = -c_1^{(n)}$. When the integration constants satisfy this condition the solution is *no longer* singular at the roots of $Q_2^{(n)}$.

¶ Note that Nickel [13] excludes the cases $k = 0$ and $k = n$.

3.2. Landau conditions for $Y^{(n)}$

Consider, now, the Landau conditions♯ for these integrals $Y^{(n)}$ which amounts to solving

$$\begin{aligned} &\alpha_j \cdot \Bigl(\Bigl(\frac{1}{2w} - \cos(\phi_j)\Bigr)^2 - 1\Bigr) \;=\; 0, \qquad j = 1, 2, \cdots, n, \\ &\beta_j \cdot \phi_j \;=\; 0, \qquad\qquad j = 1, 2, \cdots, n-1, \\ &2\,\alpha_j \cdot \Bigl(\frac{1}{2w} - \cos(\phi_j)\Bigr) \cdot \sin(\phi_j) \\ &\quad + 2\alpha_n \cdot \Bigl(\frac{1}{2w} - \cos(\phi_n)\Bigr) \cdot \sin(\phi_n) + \beta_j \;=\; 0, \quad j = 1, 2, \cdots, n-1 \end{aligned} \tag{29}$$

for α_j and β_j not all equal to zero. The Landau singularities are obtained for the following configurations. In the first configuration, $\alpha_n = 0$, one obtains the singularities $w = \pm 1/4$. The second configuration $\alpha_j \neq 0$, for all j, gives when at least one β_j is not zero, the singularity $w = 1/4$. For $\beta_j = 0$, for all j, one obtains the equations (25) for, respectively, k angles ϕ^+ and $n-k$ angles ϕ^-. The constraint (15) becomes:

$$k\,\phi^+ \,+\, (n-k)\,\phi^- \;=\; 2\pi \tag{30}$$

which is equivalent to

$$\cos\bigl(k\,\phi^+\bigr) \;=\; \cos\bigl((n-k)\,\phi^-\bigr)\,, \tag{31}$$

$$\sin\bigl(k\,\phi^+\bigr) \;=\; -\sin\bigl((n-k)\,\phi^-\bigr)\,. \tag{32}$$

Using only the first condition (31), one obtains (26) which are the singularities of the linear ODE, while using both conditions (31), (32), one obtains the polynomials corresponding to $k = 0$ and $k = n$ in (26), which *only give the singularities of the multiple integral*. The last configuration is embedded in the second configuration when $\alpha_j = 0$ for $j = 1, 2, \cdots, p$ and $\alpha_j \neq 0$ for $j = p+1, \cdots, n$. These polynomials are just given by (30) with $n-p$ instead of n, since, in the constraint (15), the angles ϕ_j, $j = 1, \cdots, p$ become equal to zero by the equations (29). The Landau conditions for $Y^{(n)}$ give then the singularities of $Y^{(p)}$, with $p = 1, 2, \cdots, n-1$. These singularities are *spurious* with respect to the linear ODE.

3.3. Kramers-Wannier duality subtleties: w versus s

In the integrals of the previous section, the calculations are performed in the variable w. In particular we have made no distinction between the high and low temperature regime (s small or large). When trying to bridge the families of "Ising class" integrals we are presenting in this paper and the actual Ising model integrals $\chi^{(n)}$, we should point out the subtleties of using the self-dual variable w which does not distinguish between the interior of the unit circle $|s| < 1$ and the exterior of the unit circle $|s| > 1$.

Let us come back to solution $Y^{(3)}$, and let us consider the singularity $1 - 5w = 0$. The solution $Y^{(3)}(w)$ is *not* singular at $w = 1/5$ where it takes a finite value (22), although the polynomial $1-5w$ still remains in the algebraic expression of the function. One has a similar situation for the singularities $1 + 3w + 4w^2 = 0$. The non "cancellation" of these polynomials, in the algebraic expression of the solution, may

♯ See the resolution of Landau conditions for the more elaborate family of integrals of Section (4).

mean that the term $1 - 16w^2$, occurring in the square root (21), is a perfect square when written in some other variable. The last is nothing else than the variable s.

The analytic continuation in the variable s is performed by writing the linear ODE corresponding to $Y^{(3)}$ in the variable s, and solving. One obtains the following solutions

$$F(s) \; = \; \mu_1 \cdot Y^{(3)}(s) \; + \; \mu_2 \cdot Y^{(3)}\Big({{1} \over {s}}\Big) \qquad \qquad \hbox{where} \qquad (33)$$

$$\begin{aligned} Y^{(3)}(s) \; &= \; {{N_Y} \over {D_Y}} \qquad \qquad \hbox{with:} \\ N_Y \; &= \; \big(6\,s^7 + s^5 - 11\,s^4 + 8\,s^3 - 32\,s^2 + 16\,s - 16\big) \cdot s^6, \\ D_Y \; &= \; \big(1 + s^2\big)^2\,(s+1)^2\,(s-1)^4\,\big(2\,s^2 + 3\,s + 2\big)\,\big(2\,s^2 - s + 2\big) \\ & \qquad \times \,\big(s^2 + s + 2\big)\,(s-2) \end{aligned}$$

The polynomials $(1 \, - 5w)$ and $(1 \, + 3\,w \, + 4\,w^2)$, written in the variable s, factorize, respectively, as $(1-2s) \cdot (s-2)$ and $(1 \, + s \, + 2\,s^2)(2 \, + s \, + \, s^2)$. The "physical" solution $Y^{(3)}(s)$ is *not* singular at the point $1 \, - 2s \, = \, 0$ and at the quadratic points $1 \, + s \, + 2\,s^2 \, = \, 0$, which are *inside the unit circle* ($|s| \, < \, 1$). Do note, however, that the "physical" solution $Y^{(3)}(s)$ *actually presents a singularity at* $s - 2 \, = \, 0$, as well as at $2 \, + s \, + \, s^2 \, = \, 0$, which are *outside the unit circle* ($|s| > 1$). The same conclusions hold for $Y^{(n)}$, $n > 3$.

Let us now present a second family of more elaborate integrals for which we address the same questions: do the Landau singularities identify with the singularities of the linear ODE or with the singularities of the integral ? Do the subtleties encoutered using the self-dual variable w instead of the s variable also occur in this family?

4. The Diagonal model

Let us first recall the fundamental, and deep, mathematical relation between Hadamard product of series, multiple integration, and diagonal reduction [20, 21]. Christol's conjecture amounts to saying that any algebraic power series in n variables is the "diagonal"of a rational power series of $2\,n$ variables [20, 22, 23, 24]. These kind of quite remarkable results [24, 25, 26, 27] gave us the idea to build our second "toy model" by considering the "diagonal" of the integrand (seen as a function of $(n-1)$ angles ϕ_j) occurring in the multiple integrals corresponding to the $\chi^{(n)}$'s, and thus perform, instead of $n-1$ integrals, *only one* integration.

At first sight, one can imagine that such a drastic reduction completely trivializes the function one tries to calculate, and yields a new function with totally different singularities. On the other hand, in view of the previous non-trivial results [20, 22, 23, 24, 25, 26, 27], one can, a contrario, imagine that such a "diagonal reduction" procedure keeps "some" relevant analytical informations, especially on the singularities. Replacing the quite involved multiple integral by an integration on *only one* angle, and since the integrand is an algebraic function of trigonometric functions of this angle and of the variable $w \, = \, s/(1+s^2)/2$, the result will be holonomic. This integral on only one variable will be solution of a finite order linear differential equation in w with polynomial coefficients. Could it be possible that the singularities of this (probably Fuchsian) linear ODE, keep some "memory" of the singularities of the involved initial multiple integral?

The second set of integrals, we present in this paper, make use of the following simplifications. From the integrand of $\chi^{(n)}$ in (3), quicking out $G^{(n)}$ and the product on y_j, we only keep the following quantity in the integrand:

$$
\frac{1}{n!} \cdot \frac{1 + \prod_{i=1}^{n} x_i}{1 - \prod_{i=1}^{n} x_i} \;\; = \;\; -\frac{1}{n!} \;\; + \frac{2}{n!} \cdot \frac{1}{1 - \prod_{i=1}^{n} x_i} \qquad (34)
$$

This is reminiscent of the assumptions made in [1] that the singularities (or a subset of them) arise from the symmetry points of the integrand and the vanishing of the denominator of the above quantity.

As previously explained, instead of performing the integration on $(n-1)$ angles ϕ_j, we integrate the above quantity on the principal diagonal, i.e. from a $(n-1)$-dimensional integral we reduce ouselves to *only one* integration angle ϕ:

$$
\phi_1 = \phi_2 = \cdots = \phi_{n-1} = \phi, \qquad\qquad \phi_n = -(n-1)\,\phi \qquad (35)
$$

Call $\Phi_D^{(n)}$ the integral on ϕ of (34):

$$
\Phi_D^{(n)} \;\; = \;\; -\frac{1}{n!} \;\; + \frac{2}{n!} \int_0^{2\pi} \frac{d\phi}{2\pi} \, \frac{1}{1 \, - x^{n-1}(\phi) \cdot x((n-1)\phi)} \qquad (36)
$$

where $x(\phi)$ is given by (6). Expression (36), when expanded in the variable x, becomes:

$$
\Phi_D^{(n)} \;\; = \;\; -\frac{1}{n!} \;\; + \frac{2}{n!} \int_0^{2\pi} \frac{d\phi}{2\pi} \sum_{p=0}^{\infty} x^{p(n-1)}(\phi) \cdot x^p((n-1)\phi) \qquad (37)
$$

The term x^p has a Fourier expansion given by [5]:

$$
x^p \;\; = \;\; w^p \cdot \Bigl(b(0,p) \;\; + 2 \sum_{k=1}^{\infty} w^k \cdot \cos k\phi \cdot b(k,p) \Bigr) \qquad (38)
$$

where $b(k,p)$ is a *non terminating* hypergeometric series that reads (with $m = k + p$):

$$
b(k,p) \;\; = \;\; \binom{m-1}{k} \times \qquad (39)
$$

$$
{}_4F_3\Bigl(\frac{(1+m)}{2}, \frac{(1+m)}{2}, \frac{(2+m)}{2}, \frac{m}{2}; 1+k, 1+p, 1+m; \, 16w^2 \Bigr)
$$

With this Fourier series, the integration rules are straigthforward.

The fully integrated expansion of $\Phi_D^{(n)}$ simply reads:

$$
\Phi_D^{(n)} \;\; = \; -\frac{1}{n!} \qquad (40)
$$

$$
+ \frac{2}{n!} \sum_{p=0}^{\infty} \sum_{k=0}^{\infty} \, w^{n(k+p)} \cdot C_k \cdot b(k,p) \cdot b\Bigl(k(n-1), p(n-1)\Bigr)
$$

where $C_0 = 1$ and $C_k = 2$ for $k \geq 1$. These sums are clearly less consuming in the computational effort as compared to the multidimensional integrals (3).

4.1. Fuchsian linear differential equations for $\Phi_D^{(n)}$ and their singularities

Generating sufficiently large series in w for each $\Phi_D^{(n)}$, we have actually been able to find the corresponding linear differential equations up to $n = 8$. The linear ODE's for

$\Phi_D^{(3)}$, and $\Phi_D^{(4)}$, are of order four and are given in Appendix D. For instance, the linear ODE satisfied by $\Phi_D^{(3)}$ reads:

$$\sum_{m=1}^{4} a_m(w) \cdot \frac{d^m}{dw^m} F(w) \;=\; 0 \qquad (41)$$

where

$$a_4(w) \;=\; (1-w)\,(1+2w)\left(1-16w^2\right)\,\left(1\,+3\,w\,+4\,w^2\right)\,w^2\cdot P_4(w)$$
$$a_3(w) \;=\; w\cdot P_3(w), \qquad a_2(w) \;=\; P_2(w), \qquad a_1(w) \;=\; P_1(w)$$

the polynomials $P_i(w)$ being displayed in Appendix D.

One does remark, on this second example, that the singularities of the linear ODE for $\chi^{(3)}$ [4, 5] actually pop out in the linear ODE of this "toy integral", *in particular the quite non trivial* $1\,+3\,w\,+4\,w^2\;=\;0$ singularities.

The linear differential equation (41) is of Fuchsian type. It has just regular singularities: one can actually see that the degree eleven polynomial P_4 does correspond to *apparent* singularities. The linear ODE's for $\Phi_D^{(3)}$ and for the next $\Phi_D^{(n)}$'s, all have logarithmic singularities at the points $w\,=\,0,\,\pm 1/4$ and $w\,=\,\infty$. The other regular singular points have a $x^{-1/2}$ singularity. The singularities $1-4w^2=0$ also carry a $x^{1/2}$ behavior. Note that the order q of the linear differential equations of these $\Phi_D^{(n)}$'s follows a simple (parity dependent) linear rule as a function of n, namely $q\;=\;(2n\,-(-1)^n\,+9)\,/4$.

Let us recall the "nickellian" singularities (1) found by Nickel for the $n-$particle contributions $\chi^{(n)}$ to the magnetic susceptibility of Ising model [1]. These singularities, written as a function of Chebyshev polynomials of first kind

$$\frac{1}{2\,w} \;=\; T_k\Bigl(\cos\Bigl(\frac{2\pi}{n}\Bigr)\Bigr) \,+\, T_m\Bigl(\cos\Bigl(\frac{2\pi}{n}\Bigr)\Bigr), \qquad 0\le k,m\le n \qquad (42)$$

can all be obtained from this "toy" model. Recalling the definition of the variable $w\;=\;s/2/(1+s^2)$ which is self dual (i.e. invariant by $s\,\rightarrow\,1/s$), one may introduce the following partition on the singularities of the linear ODE corresponding to $\Phi_D^{(n)}$. We denote by $P_1^{(n)}$ the polynomials where the roots (i.e. singularities) are given by (42). We denote by $P_2^{(n)}$ the polynomials where the roots are not given by (42) but are in $\mathcal{W}_c$. Finally, we denote by $P_3^{(n)}$ the polynomials where, at least, one root is not in $\mathcal{W}_c$. Along this partition, the polynomials for $\Phi_D^{(n)}$, $n=3,\cdots,8$, are respectively (void means there is no such type of polynomials):

$$P_1^{(3)} \;=\; (1-w)(1+2w)(1-16w^2)\cdot w, \qquad (43)$$
$$P_2^{(3)} \;=\; \text{void}, \qquad P_3^{(3)} \;=\; 1+3w+4w^2$$

for $\Phi_D^{(3)}$,

$$P_1^{(4)} \;=\; (1-4w^2)\,(1-16w^2)\cdot w, \qquad (44)$$
$$P_2^{(4)} \;=\; \text{void}, \qquad P_3^{(4)} \;=\; \text{void}$$

for $\Phi_D^{(4)}$,

$$P_1^{(5)} \;=\; (1+w)(1-16w^2)(1-3w+w^2)(1+2w-4w^2)\cdot w,$$
$$P_2^{(5)} \;=\; (1-w)(1+2w), \qquad (45)$$
$$P_3^{(5)} \;=\; (1-w-3w^2+4w^3)(1+8w+20w^2+15w^3+4w^4)$$

for $\Phi_D^{(5)}$,

$$P_1^{(6)} = (1-w^2)\,(1-16w^2)\,(1-4w^2)\,(1-9w^2)\cdot w, \qquad (46)$$
$$P_2^{(6)} = \text{void}, \qquad P_3^{(6)} = 1 - 10\,w^2 + 29\,w^4$$

for $\Phi_D^{(6)}$,

$$\begin{aligned} P_1^{(7)} &= w\cdot(1-16w^2)(1-5w+6w^2-w^3)(1+2w-8w^2-8w^3) \\ &\quad (1+2w-w^2-w^3), \\ P_2^{(7)} &= (1+w)(1-3w+w^2)(1+2w-4w^2), \qquad (47) \\ P_3^{(7)} &= (1+12w+54w^2+112w^3+105w^4+35w^5+4w^6) \\ &\quad (1-3w-10w^2+35w^3+5w^4-62w^5+17w^6+32w^7-16w^8) \end{aligned}$$

for $\Phi_D^{(7)}$,

$$\begin{aligned} P_1^{(8)} &= w\cdot(1-16w^2)(1-4w^2)(1-2w^2)(1-8w^2)(1+4w+2w^2) \\ &\quad (1-4w+2w^2), \\ P_2^{(8)} &= (1-w^2)(1-9w^2), \qquad (48) \\ P_3^{(8)} &= 1-26w^2+242w^4-960w^6+1685w^8-1138w^{10} \end{aligned}$$

for $\Phi_D^{(8)}$.

Discarding the singularities $1-16w^2 = 0$, one may remark, that for n odd, the singularities, roots of the polynomials $P_2^{(n)}$, are also roots of the polynomials $P_1^{(n-2)}$. The remark holds for n even, with the additional feature that $1-4w^2 = 0$ are common singularities to $\Phi_D^{(n)}$, n even.

4.2. Singularities of the linear ODE's versus singularities of particular solutions

Having not been able to find all the solutions of the linear differential equations satisfied by (36), one can use the connection matrix method we introduced in [8] to find, for some values of n, the singular behavior of (36) at each regular singularity.

This connection matrix method [8] was used to find the singular behavior of $\chi^{(3)}$ and $\chi^{(4)}$, the third and fourth "particle" contribution to the Ising magnetic susceptibility [8]. Let us sketch the method. The method consists in equating, at some matching points, the two sets of series corresponding, respectively, to expansions around a singular point and the nearest other singular point. The matching points have to be taken in the radius of convergence of *both* series. Connection matrices of two singularities that are not "neighbors" have to be deduced using some path of "neighboring" connection matrices.

We denote by q the order of the linear differential equation. Connecting the local series-solutions at the regular singular points $w = w_1$, and $w = w_2$, amounts to finding the $q\times q$ matrix $C(w_1, w_2)$ such that

$$\mathcal{S}^{(w_1)} \; = \; C(w_1, w_2)\cdot \mathcal{S}^{(w_2)} \qquad (49)$$

where $\mathcal{S}^{(w_1)}$ and $\mathcal{S}^{(w_2)}$ denote the vectors whose entries are the local series-solutions at respectively the singular point w_1 and w_2. These solutions are evaluated numerically at q arbitrary points around a point belonging to both convergence disks of the series-solutions. We have considered the linear differential equations corresponding to $\Phi_D^{(3)}$, $\Phi_D^{(4)}$, $\Phi_D^{(5)}$ and $\Phi_D^{(6)}$.

For the toy integral $\Phi_D^{(3)}$ whose linear differential equation has the singularities $w = 0$, $w = 1/4$, $w = -1/4$, $w = 1$, $w = -1/2$, $w = \infty$ and $w = w_1$, w_2 where w_1 and w_2 are the roots of $1+3w+4w^2 = 0$, one obtains the following singular behavior around each singularity

$$\begin{aligned}
\Phi_D^{(3)}\,(\mathrm{singular}, 1/4) \;&=\; -\frac{1}{4\pi}\;\ln(x)\cdot S_3^{(1/4)}(x),\\
\Phi_D^{(3)}\,(\mathrm{singular}, -1/4) \;&=\; -\frac{5}{8\,\pi}\cdot x\cdot \ln(x)\cdot S_3^{(-1/4)}(x),\\
\Phi_D^{(3)}\,(\mathrm{singular}, 1) &= \frac{i}{\sqrt{6}}\cdot x^{-1/2}\cdot S_1^{(1)}(x),\\
\Phi_D^{(3)}\,(\mathrm{singular}, -1/2) \;&=\; \frac{i}{\sqrt{3}}\cdot\, x^{-1/2}\cdot S_1^{(-1/2)}(x),\\
\Phi_D^{(3)}\,(\mathrm{singular}, \infty) &= \;\frac{3\,i}{4\,\pi}\cdot x\cdot \ln(x)\cdot S_3^{(\infty)}(x),\\
\Phi_D^{(3)}\,(\mathrm{singular}, w_1) &= \;b_1\cdot\, x^{-1/2}\cdot S_1^{(w_1)}(x)
\end{aligned}$$

where $S_j^{(w_s)}$ are series-solutions analytical at $x = 0$ ($x = 1 - w/w_s$ is the expansion local variable), and such that $S_j^{(w_s)}(0) = 1$. Remark the exact value of b_1, we obtain, for the singularities $1 + 3w + 4w^2 = 0$, namely $b_1 = 0$. Recalling the singularity partition (43), one sees that only the singularities given by the polynomials where all the roots are in $\mathcal{W}_c$ are actually present in the integral.

We have also considered the linear ODE for $\Phi_D^{(4)}$ which contains only singularities in $\mathcal{W}_c$. All these singularities occur in the integral. The next example is the linear ODE for $\Phi_D^{(5)}$. All the singularities, given as roots of the polynomial $P_1^{(5)}$, occur in the integral, while the singularities roots of $P_2^{(5)}$ and $P_3^{(5)}$ *do not.* Note that two complex singularities, given by the polynomial $1 - w - 3w^2 + 4w^3 = 0$, and all the roots of $1+8w+20w^2+15w^3+4w^4 = 0$ are not in $\mathcal{W}_c$. The last example we have considered is $\Phi_D^{(6)}$. Its linear differential equation has four singularities out of $\mathcal{W}_c$, namely the four roots of $1 - 10\,w^2 + 29\,w^4 = 0$. Here also, we found that only the polynomials where all the roots are in $\mathcal{W}_c$, *correspond to singularities of the integral.*

Note that the amplitudes at the singularities given by (42) can be obtained from ††:

$$\mathcal{A} \;=\; \frac{2w_s}{n\sqrt{2(n-1)}}\cdot\sum_{k,m}\frac{\sin^2\left(\frac{2\pi k}{n}\right)}{\sqrt{1-\cos\left(\frac{2\pi k}{n}\right)\cos\left(\frac{2\pi m}{n}\right)}} \qquad (50)$$

where the sum is restricted to the integers that satisfy (42) for the singularity w_s. This formula is in agreement with the results of the connection matrix method, for $n = 3, \cdots, 6$, confirming its efficiency.

The natural question now, is to find out whether *the Landau singularities identify, or have an overlap, with the singularities of the integral or, rather, with the singularities of the linear ODE.*

†† This is equivalent to the formula (14) forwarded by Nickel [1] for the $\chi^{(n)}$ and for the same type of singularities.

4.3. Singularities from Landau conditions

Before solving the Landau conditions for our integrals $\Phi_D^{(n)}$, some additional comments on Landau formulation should be underlined. The quantity x_j given in (6), is written as†

$$x_j \; = \; \frac{1}{2w} - \cos(\phi_j) \, + i \, \sqrt{1 - \Bigl(\frac{1}{2w} - \cos(\phi_j)\Bigr)^2 \, + i \, 0^+} \tag{51}$$

Define

$$\cos(\zeta) \; = \; \frac{1}{2w} - \cos(\phi), \tag{52}$$

$$\sin(\zeta) \; = \; \sqrt{1 - \Bigl(\frac{1}{2w} - \cos(\phi_j)\Bigr)^2 \, + i \, 0^+} \tag{53}$$

and, similarly, $\cos(\zeta_n)$ and $\sin(\zeta_n)$, where ϕ is changed to $(n-1)\,\phi$.

Using some identities on Chebyshev polynomials and noting that:

$$x^{n-1}(\phi) \; = \; T_{n-1}\,(\cos(\zeta)) \, + i \, \sin(\zeta) \cdot U_{n-2}\,(\cos(\zeta)) \tag{54}$$

the ratio (34), with the constraint (35), is cast in the form

$$\frac{\cos(\zeta_n) \, + T_{n-1}\,(\cos(\zeta)) \, - i \, \sin(\zeta_n) + i \, \sin(\zeta) \cdot U_{n-2}\,(\cos(\zeta))}{\cos(\zeta_n) \, - T_{n-1}\,(\cos(\zeta)) \, - i \, \sin(\zeta_n) - i \, \sin(\zeta) \cdot U_{n-2}\,(\cos(\zeta))} \tag{55}$$

Some manipulations give for the integral $\Phi_D^{(n)}$:

$$\Phi_D^{(n)} \;\; = \;\; \frac{1}{n!} \int_0^{2\pi} d\phi \cdot \frac{\sin(\zeta_n) \, - \sin(\zeta) \, U_{n-2}(\cos(\zeta))}{\cos(\zeta_n) \, - T_{n-1}(\cos(\zeta))} \tag{56}$$

where U_n is the Chebyshev polynomial of second kind. In this form the denominator of the integrand is polynomial in w and $\cos(\phi)$, resulting from a "rationalization" of the integrand. This procedure introduces the "Galois companions" of the integral and we may expect the Landau conditions to generate, up to spurious singularities, all the singularities of the linear ODE rather than those of the integral. The second comment is on the branch points appearing in the numerator. The manifolds defining these singularities should be added in the Landau conditions.

In the sequel, we give the singularities, obtained from the Landau conditions, in two cases. Firstly we allow the production of the singularities of the "Galois companions", and consider the locus of singularities of the branch points. In a second step, we restrict the analysis on the integral without considering the branch points locus.

The singularities, when the branch points appearing in the numerator are not considered in Landau conditions (see Appendix E), are the roots of:

$$T_{n-1}\Bigl(\frac{1}{2w} \, - 1\Bigr) \, + 1 \, - \frac{1}{2w} \;\; = \;\; 0 \tag{57}$$

and

$$T_{n-1}\Bigl(\frac{1}{2w} \, + 1\Bigr) \, + (-1)^{n-1} \, - \frac{1}{2w} \;\; = \;\; 0 \tag{58}$$

† $i\,0^+$ is introduced for an easy control of our computations.

and the w-polynomials obtained by eliminating the variable v from:

$$T_{n-1}\,(v)\;\;+T_{n-1}\left(\frac{1}{2w}\;-v\right)\;-\;\frac{1}{2w}\;=\;0, \qquad (59)$$

$$U_{n-2}\,(v)\;\;-U_{n-2}\left(\frac{1}{2w}-v\right)\;=\;0$$

When the branch points are included, we obtain the additional polynomials (k and m are integers), given by:

$$T_{n-1}\left(\frac{1}{2w}\;-(-1)^k\right)\;+(-1)^m\;-\;\frac{1}{2w}\;\;=\;0, \qquad (60)$$

and by the elimination of v from:

$$T_{n-1}(v)-\left(\frac{1}{2w}\;\pm 1\right)\;=\;0, \qquad T_{n-1}\left(\frac{1}{2w}\;-v\right)\;\pm 1\;=\;0 \qquad (61)$$

Remarkably we find that the roots of the polynomials (57), (58) and (59) *identify with the singularities of the differential equations governing* (36) up to $n\,=\,8$, i.e. with the singularities given in the polynomials $P_1^{(n)}$, $P_2^{(n)}$ and $P_3^{(n)}$. The check has been extented up to $n=16$ with the singularities of *the linear differential equations obtained, this time, modulo some primes.*

Note that the polynomials given by the first set (57) have all their roots in $\mathcal{W}_c$, and similarly for the second set (58) for n even.

The polynomials (60), (61) give additional singularities not given by (57-59). For $n=3,4,5$, they read:

$$n\;=\;3, \qquad (1+3w)(1-5w)(1-w-4w^2)$$

$$n\;=\;4, \qquad \left(1\pm 3\,w-w^2\right)(1\,\pm 6\,w\;+8\,w^2\;\pm 4\,w^3)$$

$$n\;=\;5, \qquad (1+3\,w)\left(4\,w^3+3\,w^2-w-1\right)\left(5\,w^2+5\,w+1\right)$$

$$\left(1-16\,w^2-2\,w^3+56\,w^4\;-16\,w^5-63\,w^6+8\,w^7+16\,w^8\right)$$

$$\left(-1+8\,w-20\,w^2+17\,w^3\right).$$

These singularities do not appear in the corresponding linear differential equation, they are *spurious*.

Recall that these "spurious" singularities (with respect to the linear ODE) are obtained when the manifold of the branch points appearing in the numerator of (56) are considered in the Landau conditions. Let us now, exclude these (spurious) conditions on the branch points and, furthermore, let us control the solutions introduced by the normalization procedure in order to extract the "physical" Landau singularities¶ from the "Galois" Landau singularities (see Appendix E). In this case, the singularities are obtained as roots of the gcd of the polynomials:

$$T_{n-1}\left(\frac{1}{2w}\;-1\right)\;+1\;-\;\frac{1}{2w}\;\;=\;0, \qquad (62)$$

$$1+U_{n-2}\left(\frac{1}{2w}-1\right)\;=\;0$$

and

$$T_{n-1}\left(\frac{1}{2w}\;+1\right)\;+(-1)^{n-1}\;-\;\frac{1}{2w}\;\;=\;0, \qquad (63)$$

¶ By "physical", we mean "concerns" the integral, i.e. a particular solution of the linear ODE.

$$\sqrt{1 - \left(\frac{1}{2w} - (-1)^{n-1}\right)^2} + \sqrt{1 - \left(\frac{1}{2w} + 1\right)^2}\, U_{n-2}\left(\frac{1}{2w} + 1\right) \; = \; 0$$

and the w-polynomials obtained by eliminating the variable v from

$$T_{n-1}(v) \; + T_{n-1}\left(\frac{1}{2w} - v\right) - \frac{1}{2w} \; = \; 0, \qquad (64)$$

$$U_{n-2}(v) \; - U_{n-2}\left(\frac{1}{2w} - v\right) \; = \; 0,$$

$$\sqrt{1 - \left(\frac{1}{2w} - T_{n-1}(v)\right)^2} + \sqrt{1 - \left(\frac{1}{2w} - v\right)^2} \cdot U_{n-2}\left(\frac{1}{2w} - v\right) \; = \; 0$$

We see that we obtain the same set of polynomials (57), (58) and (59) dressed with constraints. The singularities, obtained from these conditions, correspond to the roots of the following polynomials

$$\begin{aligned}
n \,&=\, 3, \qquad && (1-4w)(1+2w)(1-w),\\
n \,&=\, 4, \qquad && (1-4w)(1+4w)(1+2w)(1-2w),\\
n \,&=\, 5, \qquad && (1-4w)(1+w)(1+2w-4w^2)(1-3w+w^2),\\
n \,&=\, 6, \qquad && (1-4w)(1+4w)(1-w^2)(1-9w^2)(1-4w^2),\\
n \,&=\, 7, \qquad && (1-4w)(1-5w+6w^2-w^3)(1+2w-8w^2-8w^3)\\
& && (1+2w-w^2-w^3),\\
n \,&=\, 8, \qquad && (1-4w)(1+4w)(1-2w^2)(1-8w^2)(1-4w^2)(1-9w^2)\\
& && (1-4w+2w^2)(1+4w+2w^2)
\end{aligned}$$

which are exactly (up to $1+4w$ for n odd) the roots given by $P_1^{(n)}$, i.e. Nickel's for the Ising model (42). All these conditions (62), (63) and (64) give the "nickellian singularities" obtained from the simple form given in (42).

Recall that we know exactly, by the results of the connection matrix method [8] given in the previous section, which singularities are actually present in the integrals for $n = 3, 4, 5$ and $n = 6$, and, especially, the *non occurrence* in the integrals of the singularities (not in $\mathcal{W}_c$) roots of $P_3^{(3)}$ for $n = 3$, roots of $P_2^{(5)}$, $P_3^{(5)}$ for $n = 5$ and roots of $P_3^{(6)}$ for $n = 6$. So, and at least for $n = 3, 4, 5$ and $n = 6$, the singularities obtained as roots of the polynomials (62), (63) and (64) are in complete agreement with the singularities of the integrals.

We thus see that, depending on how the Landau conditions are performed, the Landau singularities may be a set larger than the singularities of the linear ODE, or identify with the singularities of the ODE, or, even, could be the singularities of the integral.

4.4. The singularities of the linear ODE's: an involved set with a structure

Assuming that the polynomials (57), (58) and (59) do indeed reproduce the singularities of the linear ODE corresponding to the integrals $\Phi_D^{(n)}$ for $n > 16$, one may use these formula to see how the singularities accumulate in the complex plane of the variable s, as n goes larger. Figure 1 shows the loci of the singularities outside the unit circle $|s| = 1$.

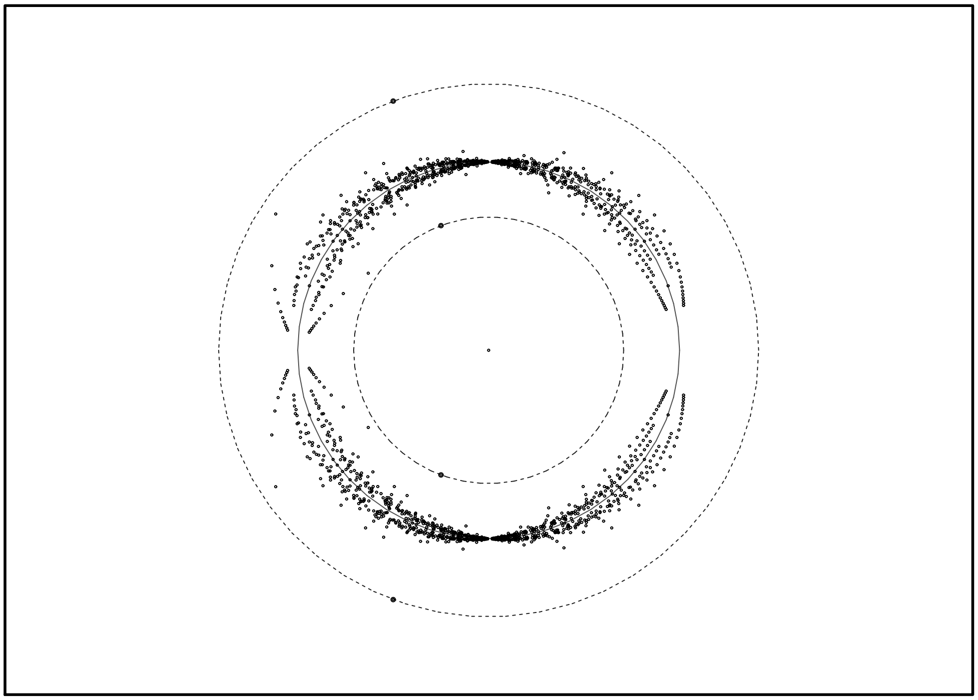

Fig. 1 The roots of $P_3^{(n)}$ in the complex plane of the variable s, up to $n = 47$.

These singularities with $|s| \neq 1$ come only from the polynomials (58) for n odd, and from the polynomials (59). Note that the singularities from (58) *are all in the left half-plane of the variable* s. The concentric circles are respectively (from the inner to the outer) $|s| = 1/\sqrt{2}$, $|s| = 1$ and $|s| = \sqrt{2}$. The outer concentric circles correspond to the modulus in s of the quadratic numbers $1 + 3w + 4w^2 = 0$ (that read in the variable s, $(1 + s + 2s^2)(2 + s + s^2) = 0$). We note the remarkable fact that all the singularities are lying in the annulus defined by these inner and outer circles and are far from the singularities $s = \pm 1$. These last two points seem to be accumulation points to which the singularities are converging as n goes higher.

The proliferation of the singularities out of the unit circle $|s| = 1$ is obviously polynomial. In fact, the degree of the polynomials $P_3^{(n)}$ follows the simple rule

$$\deg\Big(P_3^{(n)}\Big) \, = \, \frac{1}{8}\left(2n^2 - 15\right) - \frac{(-1)^n}{8}\,(4n+1)\,, \qquad n = 3, 4, \cdots$$

When "all" the solutions of the ODE for each n are considered, their singularities form the unit circle $|s| = 1$, and the structure around, shown in Fig. 1.

4.5. More Kramers-Wannier duality subtleties: analytical continuation in s

Let us consider, now, the linear differential equations corresponding to the integrals $\Phi_D^{(n)}$, which, written in the variable s, are obviously covariant by Kramers-Wannier duality $s \leftrightarrow 1/s$. Note that, this *does not mean* that the solutions of these self-dual differential equations are invariant by Kramers-Wannier duality $s \leftrightarrow 1/s$. Consider first, the simplest integral of this family, namely $\Phi_D^{(2)}$. As a function of the variable

w it is a solution of a third order differential operator which is the direct sum of two linear differential operators:

$$Dw \, \oplus \Bigl(Dw^2 \, + \frac{1\, - 48\, w^2}{(1 - 16\, w^2)\, w} \cdot Dw \, - \frac{16}{1 - 16\, w^2} \Bigr) \tag{65}$$

with general solution

$$c_0 \, + c_1 \cdot K(4w) \, + c_2 \cdot CK(4w) \tag{66}$$

where (CK) and K are respectively the (complementary) complete elliptic integral of first kind:

$$CK(4w) \, = \, K\Bigl(\sqrt{1 - 16\, w^2} \Bigr) \tag{67}$$

$$K(4w) \, = \, {}_2F_1\Bigl([1/2, 1/2], [1], 16\, w^2 \Bigr) \tag{68}$$

The linear differential operator (65), written in the variable s, gives a third order linear differential operator $L_3(s)$ which reads:

$$L_3(s) \, = \, Ds \, \oplus L_2(s) \qquad \text{where,}$$

$$L_2(s) \, = \, Ds^2 \, + \frac{s^4 + 4\, s^2 \, - 1}{(s^4 \, - 1)\, s} \cdot Ds \, - \frac{4}{(1 + s^2)^2} \tag{69}$$

The third order linear differential operator $L_3(s)$ has the following general solution:

$$c_0 \, + c_1 \cdot (1 + s^2) \cdot K(s^2) \, + c_2 \cdot \Bigl(1 + s^2\Bigr) \cdot CK \left(s^2\right) \tag{70}$$

The corresponding linear differential operators associated with the $\Phi_D^{(n)}$ have the constant fonction as solution. Note that, for the integrals $\Phi_D^{(n)}$, this constant♯ can actually be calculated and is equal to 1/2.

The particular solution $\Phi_D^{(2)}(s)$ reads:

$$\Phi_D^{(2)}(s) \, = \, \frac{1}{2} + \, \frac{1}{2}(1 + s^2) \cdot K(s^2) \tag{71}$$

The general solution of $L_2(s)$ can be written as:

$$\mu_1 \cdot \Phi_D^{(2)}(s) \, + \, \mu_2 \cdot \Phi_D^{(2)} \left(\frac{1}{s} \right) \tag{72}$$

The two solutions of $L_2(s)$ are *actually the same function* for, respectively, the argument s and $1/s$. Note that, this can be easily seen on the *formal solutions* of $L_2(s)$ which take the same form

$$\Phi_D^{(2)}(u) - 1/2,$$

$$\Bigl(\Phi_D^{(2)}(u) - 1/2\Bigr) \cdot \ln(u) \, + \frac{1}{16} \left(u^4 + u^6 + \frac{21}{32} u^8 + \, \cdots \right) \tag{73}$$

where $u = \, s$ at the singular point $s = \, 0$ and $u = \, 1/s$ at the point at the infinity.

The two quantities $\Phi_D^{(2)}(s)$ and $\Phi_D^{(2)}(1/s)$ are two different solutions of the self-dual differential equations, in complete analogy with $Y^{(3)}(s)$ and $Y^{(3)}(1/s)$ of Section (3.3), especially their singularities which clearly break $s \leftrightarrow 1/s$ symmetry.

We have here the exact equivalent of the situation we had [6] with the order-two linear differential equation corresponding to $\tilde{\chi}^{(2)}$. The latter reads:

$$\tilde{\chi}^{(2)} \, = \, 4\, w^4 \cdot {}_2F_1\Bigl([5/2, 3/2], [3], 16\, w^2 \Bigr) \tag{74}$$

♯ We discard from now on, the factor $1/n!$ in the definition of the integrals $\Phi_D^{(n)}$.

while the second independent solution [6] reads in terms of the MeijerG function [28]:

$$\tilde{\mathcal{S}}_3(w) \;=\; \frac{\pi}{2}\,\mathrm{MeijerG}\left([[], [1/2, 3/2]], [[2, 0], []], 16w^2\right) \qquad (75)$$

Rewriting the order-two linear differential equation in the variable s, one obtains the general solutions:

$$\begin{aligned} F(s) \;=\;& c_1 \cdot \frac{1}{s^4} \cdot \left(1 + \frac{1}{s^2}\right) \cdot {}_2F_1\left([5/2, 3/2], [2], \frac{1}{s^4}\right) \\ &\qquad + c_2 \cdot s^4 \cdot \left(1 + s^2\right) \cdot {}_2F_1\left([5/2, 3/2], [2], s^4\right) \\ =\;& c_1 \cdot \tilde{\chi}^{(2)}(1/s) \, + c_2 \cdot \tilde{\chi}^{(2)}(s) \end{aligned} \qquad (76)$$

The compatibility of (74) and (76) corresponds to remarkable identities like:

$$\begin{aligned} &{}_2F_1\left([5/2, 3/2], [3], \frac{4\, s^2}{(1+s^2)^2}\right) \;= \qquad (77)\\ &\qquad =\; (1 + u^2)^5 \,\cdot\, {}_2F_1([5/2, 3/2], [2],\, u^4) \end{aligned}$$

or

$$\begin{aligned} &{}_2F_1\left([1/2, 1/2], [1], \frac{4\, s^2}{(1+s^2)^2}\right) \;= \qquad (78)\\ &\qquad =\; (1 + u^2) \,\cdot\, {}_2F_1([1/2, 1/2], [1],\, u^4) \end{aligned}$$

where $u = s$ for small values of s and $u = 1/s$ for large values of s.

The previous calculations deal with simple linear ODE's and simple enough functions. However, for solutions of Fuchsian linear differential equations of higher order it becomes a wishful thinking to recognize, in the whole s-complex plane, that two functions are the "same function" for, respectively, the arguments s and $1/s$.

Consider, for instance, the integral $\Phi_D^{(3)}(s)$. The order four linear differential operator for $\Phi_D^{(3)}(s)$ has the following direct sum decomposition:

$$Ds \oplus \mathcal{L}_3(s) \;=\; Ds \oplus \Big(\mathcal{L}_2(s) \cdot \mathcal{L}_1(s)\Big) \qquad (79)$$

where the order-one, and order-two, linear differential operators are given in Appendix F.

The integral $\Phi_D^{(3)}(s)$ is the sum of the constant $1/2$ and of a particular solution of $\mathcal{L}_3$:

$$S(\mathcal{L}_3) \;=\; c_1 \cdot S(\mathcal{L}_1) \; + S(\mathcal{L}_1) \cdot \int ds\, \frac{S(\mathcal{L}_2)}{S(\mathcal{L}_1)} \qquad (80)$$

where $S(\mathcal{L}_1)$ is the solution of the self-dual order-one linear differential operator $\mathcal{L}_1$:

$$S(\mathcal{L}_1) \;=\; \frac{\left(1+s^2\right)\left(2+s+2\,s^2\right)}{\sqrt{1+s+s^2}\,\sqrt{2-s+2\,s^2}\,\sqrt{2+s+s^2}\,\sqrt{1+s+2\,s^2}} \qquad (81)$$

The general solution of the order-two linear differential operator $\mathcal{L}_2$ is a linear combination of $S_1(\mathcal{L}_2)$ and $S_2(\mathcal{L}_2)$ given by:

$$\begin{aligned} S_1(\mathcal{L}_2)(s) \;=\;& \frac{1}{D_0} \cdot \Big(D_K \cdot K(s^2) + D_E \cdot E(s^2)\Big), \qquad (82)\\ S_2(\mathcal{L}_2)(s) \;=\;& \frac{1}{s^2} \cdot S_1(\mathcal{L}_2)(1/s) \end{aligned}$$

where D_0, D_K and D_E are polynomials in s given in Appendix F, and $E(s^2)$ is the complete elliptic integral of second kind. The order-two linear differential operator $\mathcal{L}_2$ is not self-dual, it is self-dual "up to s^2" (see (82)).

To see that $\Phi_D^{(3)}(s)$ and $\Phi_D^{(3)}(1/s)$ are two different solutions of the linear differential operator (79), the integration in (80) has to be worked out.

As far as the singular behavior of solutions of finite order linear differential operators with polynomial coefficients is concerned, the connection matrix method we introduced in [8] *actually provides an answer to these difficult problems of analytical continuation in the whole s-complex plane*. For the singular behavior of $\Phi_D^{(3)}(s)$ at the points $(2+s+s^2)(1+s+2s^2) = 0$, we have, typically, to compose three 3×3 connection matrices linking the point $s = 0$ to $s_1 = -1/4+i\sqrt{7}/4$, root of $1+s+2s^2 = 0$ which is inside the unit circle, then to $s = i$, then to $s_2 = -1/2+i\sqrt{7}/2$, root of $s^2+s+2 = 0$ which is outside the unit circle. The corresponding singular behavior are:

$$
\begin{aligned}
\Phi_D^{(3)}\,(\mathrm{singular}, s_1) =&\; b_1 \cdot x^{-1/2} \cdot S_1^{(s_1)}(x), \\
\Phi_D^{(3)}\,(\mathrm{singular}, s_2) =&\; b_2 \cdot x^{-1/2} \cdot S_2^{(s_2)}(x)
\end{aligned} \qquad (83)
$$

where $S_j^{(s_j)}$ are series-solutions, analytical at $x = 0$ (x is the expansion local variable $x = 1 - s/s_i$), and such that $S_j^{(s_j)}(0) = 1$. The amplitudes at the singularities are $b_1 = 0$ and $b_2 = (1+i)\cdot 7^{-1/4}/2$. This means that the integral $\Phi_D^{(3)}$ is *not* singular at the two quadratic roots $1+s+2s^2 = 0$ which are inside the unit circle ($|s| < 1$). In contrast $\Phi_D^{(3)}$ actually exhibits a singular behavior at the two quadratic roots $2+s+s^2 = 0$ which are outside the unit circle ($|s| > 1$).

We have also considered the singular behavior of $\Phi_D^{(5)}$ in the complex plane of the variable s. The singularities where at least one is out of the unit circle ($|s| \neq 1$) are given by the roots of the polynomial $P_3^{(5)}$ which decomposes in two polynomials reading, respectively, $(4-2s+9s^2-2s^3+9s^4-2s^5+4s^6)$ and $(4+6s+7s^2+5s^3+2s^4)(2+5s+7s^2+6s^3+4s^4)$. The integral $\Phi_D^{(5)}$ is not singular at the roots of the first polynomial and at the roots of $(2+5s+7s^2+6s^3+4s^4) = 0$, while its analytic continuation is *actually singular* at the roots of $(4+6s+7s^2+5s^3+2s^4) = 0$ which are exterior to the unit circle ($|s| > 1$). For $\Phi_D^{(6)}$, the ODE carries as singularities out of the unit circle, the roots of $1-10w^2+29w^4 = 0$ which reads in the variable s, $16+24s^2+45s^4+24s^6+16s^8 = 0$. The integral $\Phi_D^{(6)}$ is *not singular* at these points.

As a result, one may imagine that the singularities of the $\Phi_D^{(n)}(s)$, defined in the whole complex plane of the variable s by analytical continuation of high temperature $s-$series (valid for $|s| < 1$), actually correspond to the "nickellian singularities" (1), *together with some points in the "cloud" of singularities* of Fig. 1 outside the unit circle ($|s| > 1$).

In view of these results on the integrals $Y^{(n)}$ and $\Phi_D^{(n)}$, it is necessary to revisit, *in the s variable*, the connection matrix computation for the third contribution to the susceptibility of Ising model, $\tilde{\chi}^{(3)}$ given in [8].

We find that, similarly, to our toy integral $\Phi_D^{(3)}$, $\tilde{\chi}^{(3)}$ is *not singular* at the two roots $1+s+2s^2 = 0$, for which $|s| < 1$, while its analytical continuation is *actually singular* at the two roots $2+s+s^2 = 0$, for which $|s| > 1$. The corresponding singular behavior reads:

$$
\begin{aligned}
\tilde{\chi}^{(3)}\,(\mathrm{singular}, s_1) \;=&\; b_1 \cdot S_1^{(s_1)}(x)\cdot x\cdot \ln(x), \\
\tilde{\chi}^{(3)}\,(\mathrm{singular}, s_2) \;=&\; b_2 \cdot S_2^{(s_2)}(x)\cdot x\cdot \ln(x)
\end{aligned}
$$

where $S_j^{(s_j)}$ are series-solutions, analytical at $x = 0$ (x is the local variable $x = 1-s/s_i$) and such that $S_j^{(s_j)}(0) = 1$. The amplitudes at the singularities are $b_1 = 0$ and $b_2 = (181\sqrt{7} - 7\,i)/256\pi$.

The three-particle contribution [8] to the susceptibility of Ising model, $\chi^{(3)}(s)$, is fundamentally a *high-temperature quantity*. The three-particle contribution, $\chi^{(3)}(s)$, together with $\chi^{(3)}(1/s)$, are *two different solutions* of the self-dual linear differential operator for $\chi^{(3)}(s)$.

Similarly to the $\Phi_D^{(n)}(s)$, one may imagine that the singularities of the $\chi^{(n)}(s)$, for n odd, defined in the whole complex plane of the variable s by analytical continuation of high temperature $s-$series (valid for $|s| < 1$), actually correspond to the "nickellian singularities" (1), *together with a similar "cloud" of singularities* as in Fig. 1 outside the unit circle ($|s| > 1$).

4.6. Analytic continuation of Landau conditions in the variable s

These Kramers-Wannier duality subtleties, and the breaking of the $s \leftrightarrow 1/s$ symmetry, can also be seen on the Landau conditions. Consider for instance, the polynomial $1 + 3w + 4w^2 = 0$ which appears in the linear ODE for $\Phi_D^{(3)}$ from the condition (58) and which does not occur in the integral due the constraint on this condition given in (63).

When one switches to the variable s, the polynomial $(1 + 3w + 4w^2)$ becomes $(1 + s + 2s^2)(2 + s + s^2)$. The constraint in (63) gives:

$$\sqrt{-\frac{(1+s^2)\,(s-1)^2}{s^2}} \cdot s \qquad\qquad (84)$$
$$+\; 2\sqrt{-\frac{(1+s^2)\;(s+1)^2}{s^2}} \cdot (1+s+s^2) \;=\; 0$$

The series expansions of the left-hand side of the algebraic condition (84) identify with the series expansions of:

$$\frac{\sqrt{1+s^2}}{s} \cdot (1+2\,s) \cdot \big(2+s+s^2\big) \;=\; 0 \qquad (85)$$

and

$$\frac{\sqrt{1+s^2}}{s} \cdot (s+2)\,\big(2\,s^2+s+1\big) \;=\; 0 \qquad (86)$$

for respectively, small values of s and large values of s. The gcd in (63) written in s, and for small values of s, then gives the two quadratic roots $2 + s + s^2 = 0$ as singularities of the integral $\Phi_D^{(3)}$. These singularities are in the exterior of the unit circle ($|s| > 1$). Similarly, had the integral $\Phi_D^{(3)}$ been defined for the large values of s, the gcd gives, as singularities, the roots of the polynomial $2\,s^2 + s + 1 = 0$, which are in the interior $|s| < 1$ of the unit circle.

5. Comments on the resolution of Landau conditions

In Section (2), we have recalled some basic general ideas and definitions on the Landau conditions. We have used these conditions on our two families of integrals, for which, we knew, from the outset, the set of singularities occurring in the linear ODE and

in the integral. In practice the calculations can be slightly more subtle, and require cautious††, sometimes tricky and involved, analysis on one (resp. several) complex variable(s). For instance, the condition (32) in Section (3.2) which rules out all the singularities not occurring in the integral, has to be worked out carefully to be able to control the $\pm$ signs floating around.

The idea of the Landau conditions is to get some candidates for the singularities of some (multiple) integral from simple enough (algebraic) calculations on the integrand. In practice when the integrands are not rational expressions but algebraic ones, the algebraic character of the integrand introduces some ambiguity for each of the branchs cuts.

For the calculations where the algebraic expressions are quite involved, the control of the Riemann sheet we stay on, may be quite tedious, sometimes hopeless. The algebraic integrand being solution of some polynomial, possibly of large degree, a possible approach amounts to considering all these roots together, namely the integrand of the actual "physical" (multiple) integral we are interested in, together with all his Galois "companions". Recall that, as a consequence of the algebraic character of these integrands, the (multiple) integral we are analyzing is holonomic. It is a solution of a finite order linear differential equation with polynomial coefficients, and it is therefore, canonically associated with its "Galois companions" with respect♯ to the linear ODE, namely the other solutions of the linear ODE.

The calculations of the Landau conditions can now be performed for all these roots, leading to a larger set of Landau singularities that are probably good candidates for, not only the "physical" (multiple) integral we are interested in, but also for other mathematical expressions corresponding to (multiple) integral of his Galois "companions" integrands.

The conditions, we have used to settle the set of singularities occurring in the integrals from the full set of singularities of the linear ODE, appear in (31), (32) for the first family, and in (103), (104) for the second family. Note that (103) is the denominator appearing in (56) obtained with a "rationalization procedure", introducing the "Galois companions" of the integrand, and thus leading to the singularities of the linear ODE. We are aware that, for many examples (not of the Ising class we are interested in), the resolution of Landau conditions may be more tricky and/or more subtle.

Let us close this section by a comment on the w versus s analytic continuation. We have used the self-dual variable $w = s/2/(1+s^2)$ in our previous works [4, 5, 6, 8] on the susceptibility of Ising model since it appears naturally in the integrand (3) and, especially, it is much simpler as far as computational efforts are concerned. The differential equation for $\chi^{(3)}$ has been obtained [4] with a series up to w^{359}. In the variable s, a series up to s^{699} would have been needed. When the distinction between the regimes of large values of s, and small values of s, is required, we have seen some of the subtleties that pop out. For the $Y^{(n)}$ family, this subtlety appeared through a perfect square mechanism in the complex variable s. For the $\Phi^{(n)}$ family, an equivalent mechanism of factorization appeared and the result depends on the regime considered (small s, large s). Note that this may not be the only mechanism that can be at work. One may imagine, for instance, a function, like the difference between the right-hand-side and the left-hand-side of (77), that evaluates differently according to the region

††See for instance the examples in Itzykson and Zuber's book [19], or in Eden *et. al* [9].
♯ The Galois group of permutation of the roots of the polynomial is replaced here by the differential Galois group of the linear differential equation.

in the complex plane of the variable s we are considering.

6. Towards Ising model integrals

The two families of integrals presented in the previous sections are very rough approximations to the integrals (3). The first family considered the product on y_i, integrated on the whole domain of integration of the ϕ_i. Here, we found a set of singularities occurring in the $\chi^{(n)}$ and the quadratic polynomial $1 + 3w + 4w^2 = 0$. The second family was constructed, besides discarding the factor $G^{(n)}$ and the product on y_i, by restricting the domain of integration on the principal diagonal of the angles ϕ_i. This resulted in a remarkable "memory" of the original problem since all the singularities (42), forwarded by Nickel [1, 2], are reproduced, even the quadratic roots of $1+3w+4w^2 = 0$ found [4, 5] for the linear ODE of $\chi^{(3)}$.

The following steps will be to incorporate, gradually, the various factors appearing in (3). One may, for instance, continue to discard the factor $G^{(n)}$, but incorporate the product on y_j to the $\Phi_D^{(n)}(w)$.

In this move towards the n-particle contributions $\chi^{(n)}$ of the Ising model susceptibility, it is natural to consider the following family of integrals

$$\Phi_H^{(n)} \;=\; \frac{1}{n!} \cdot \Big(\prod_{j=1}^{n-1} \int_0^{2\pi} \frac{d\phi_j}{2\pi} \Big) \Big(\prod_{j=1}^{n} y_j \Big) \cdot \frac{1 \, + \prod_{i=1}^{n} \, x_i}{1 \, - \prod_{i=1}^{n} \, x_i} \tag{87}$$

which amounts to getting rid of the $(G^{(n)})^2$ in (3).

One may restrict this multidimensional integral to a simple integral on one angular variable in the same way we did it for our previous "diagonal toy model" in Section (4):

$$\begin{aligned} I_D^{(n)} \;=\; & \frac{1}{n!} \int_0^{2\pi} \frac{d\phi}{2\pi} \times \\ & y^{n-1}(\phi) \cdot y((n-1)\phi) \cdot \frac{1 \, + \, x^{n-1}(\phi) \cdot x((n-1)\phi)}{1 \, - x^{n-1}(\phi) \cdot x((n-1)\phi)} \end{aligned} \tag{88}$$

For this last family, one obtains for $n = 3$, the singularities found for the linear ODE of the diagonal model integral $\Phi_D^{(3)}$ of Section (4) and also the singularities $(1 + 3w)(1 - 5w) = 0$. These singularities emerge as a consequence of inclusion of the y_i. Recall that these last singularities do not occur [4] in the linear ODE corresponding to $\chi^{(3)}$. Do they disappear in $\chi^{(3)}$ as a consequence of the enlargment of multidimensional integration domain, or as a consequence of the inclusion of the $G^{(n)}$ factor ?

Note that, as we move closer to the original integrand and integral (3), for instance considering the multidimensional integral (87), the calculations [4, 5, 6] (series expansions, search of the linear ODE) become much harder. Nevertheless, an exact knowledge of the singularities of the linear ODE's can be achieved using a brand new strategy that amounts to getting very large series for these integrals *modulo primes*, and in a second step, get the corresponding linear ODE's *also modulo primes*¶. Ideally,

¶ This can be seen as a *new method for obtaining exact results in lattice statistical mechanics*, and, in any case, extremely large series expansions. In forthcoming publications with A. J. Guttmann, I.Jensen and B. Nickel we will detail such "extreme" calculations of lattice statistical mechanics *modulo the primes* providing more than 1600 coefficients of the high and low temperature series expansions of the susceptibility χ of the square Ising model, as well as more than 2000 coefficients for $\chi^{(5)}$, and, *for given primes*, with 6000 coefficients.

for a large enough set of such calculations modulo different primes, one can get (from a Chinese remainder procedure) the exact linear ODE we are seeking for. In the case where the number of primes and calculations modulo these primes are not numerous enough to build the linear ODE, they are sufficient to be sure that a certain polynomial (like the ones we displayed in Section (4)) actually occurs in the head polynomial of the linear ODE, or discard its occurrence.

7. Conclusion

We have designed two families of "Ising class integrals" ($Y^{(n)}(w)$, $\Phi_D^{(n)}(w)$), for which we obtained the Fuchsian linear differential equations for large enough values of n. Each Fuchsian linear differential equation provides a set of singularities. We have used a direct resolution (for the first family of integrals) and our connection matrix method [8] (for the second family of integrals) to obtain the subset of singularities occurring in the integrals. For both families of integrals, we solved the Landau conditions and found that the singularities we obtained may identify, or *even extend*, the singularities of the linear ODE or the integrals. While it is obvious that the linear ODE, which has this holonomic integral as solution, gives all other solutions with their singular behavior, it is remarkable, as far as the locus of singularities is concerned, that the Landau conditions calculations on the integrand give informations on the other singularities not carried by the integral.

For these "Ising class integrals", we found that each family of integrals is singular in the domain $\mathcal{W}_c$. Our integrals are defined for small values of the variable w and elsewhere by analytical continuation. This variable $w = s/2/(1+s^2)$ behaves in an equal footing for small and large values of the variable s. Switching to this last s variable, we showed that the analytical continuations (from small values of s for which the integrals are defined) of these two families of integrals are singular on points occurring on the unit circle $|s| = 1$, *but also* on some points in the exterior of this circle ($|s| > 1$).

In view of these results on our toy integrals, we revisited the singular behavior of the third contribution to the magnetic susceptibility $\tilde{\chi}^{(3)}$ of the Ising model. We also found here that $\tilde{\chi}^{(3)}$, defined as series expansions for small values of the variable s and analytically continued to large values of s, is *actually singular* outside the unit circle, $|s| > 1$, at the points $2 + s + s^2 = 0$. If a similar scenario remains valid for the other $\chi^{(n)}(s)$, for n odd, one may imagine that the singularities of the $\chi^{(n)}(s)$, for n odd, defined in the whole complex plane of the variable s by analytical continuation of high temperature $s-$series (valid for $|s| < 1$), actually correspond to the "nickellian singularities" (1), *together with a similar "cloud" of singularities* as in Fig. 1 *outside* the unit circle ($|s| > 1$).

Our two families of integrals are introduced to mimic the $n-$particle contribution $\chi^{(n)}$ to the susceptibility of the square Ising model. In a forthcoming publication we will see to what extend these two families of integrals, and similar toy models, give information on the singular behavior of these integrals in the complex plane of the variable s and on the singularities that might occur in the $\chi^{(n)}$.

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8. Appendix A

We give in this Appendix the order six differential equation corresponding to

$$\tilde{Y}^{(3)}(w) \;=\; \int_0^{2\pi} \frac{d\phi_1}{2\pi} \int_0^{2\pi} \frac{d\phi_2}{2\pi} \cdot y_1\, y_2\, y_3 \qquad (89)$$

that reads ($F(w) = \tilde{Y}^{(3)}(w)$)

$$\sum_{n=0}^{6} a_n(w) \cdot \frac{d^n}{dw^n} F(w) \;=\; 0 \qquad (90)$$

where

$$\begin{aligned}
a_6(w) = &\; (1-5\,w)\,(1+3\,w)\left(1+3\,w+4\,w^2\right)(1+4\,w)^2\,(1-4\,w)^4 \\
&\; (1-w)\; w^6 \cdot P_6(w) \\
a_5(w) = &\; 2\;(1+4\,w)\,(-1+4\,w)^3\; w^5 \cdot P_5(w) \\
a_4(w) = &\; 2\,(1-4\,w)^2\,w^4 \cdot P_4(w), \quad a_3(w) = \,-4\,(1-4\,w) \cdot w^3 \cdot P_3(w) \\
a_2(w) = &\; 4\,w^2 \cdot P_2(w), \qquad a_1(w) = \; 8\,w \cdot P_1(w), \qquad a_0(w) = \; P_0(w)
\end{aligned}$$

with

$$\begin{aligned}
P_6(w) = &\; 464889446400\,w^{17} - 5967097036800\,w^{16} + 2518865879040\,w^{15} \\
&+ 1205147090944\,w^{14} - 209927829504\,w^{13} - 23192809472\,w^{12} \\
&- 94576000000\,w^{11} + 15333496352\,w^{10} + 6474551520\,w^9 + 434664056\,w^8 \\
&- 768760124\,w^7 + 47392381\,w^6 + 24032176\,w^5 - 3740777\,w^4 \\
&+ 106680\,w^3 + 7443\,w^2 - 364\,w + 49 \\
P_5(w) = &\; 5020806021120000\,w^{24} - 67498529469235200\,w^{23} \\
&+ 30557536675430400\,w^{22} + 38088998121308160\,w^{21} \\
&- 4237195746729984\,w^{20} - 9069732118712320\,w^{19} - 2846232832169984\,w^{18} \\
&+ 241117241392640\,w^{17} + 618725352473088\,w^{16} + 201166769452896\,w^{15} \\
&- 61245521268224\,w^{14} - 22970672645976\,w^{13} + 1946216478788\,w^{12} \\
&+ 1256316716689\,w^{11} - 2295775019\,w^{10} - 38685730792\,w^9 \\
&+ 7182763949\,w^8 - 2662652062\,w^7 + 133819572\,w^6 + 107083800\,w^5 \\
&- 16853044\,w^4 + 499141\,w^3 + 34119\,w^2 - 784\,w + 147 \\
P_4(w) = &\; 56233027436544000\,w^{26} - 783626482671943680\,w^{25} \\
&+ 3202063539476889600\,w^{24} + 3620634209723351040\,w^{23} \\
&- 548610793223356416\,w^{22} - 626862114022522880\,w^{21} \\
&- 294517075356614656\,w^{20} + 30188250588135424\,w^{19} \\
&+ 71955514644985856\,w^{18} + 6645859876651520\,w^{17} \\
&- 7980786065196160\,w^{16} - 956452174732544\,w^{15} \\
&+ 546200976422432\,w^{14} + 127674138396992\,w^{13} \\
&- 45199723462020\,w^{12} - 9028528613608\,w^{11} + 2584065207968\,w^{10} \\
&+ 428924564347\,w^9 - 59295462952\,w^8 - 29452365937\,w^7 \\
&+ 3600235371\,w^6 + 631583771\,w^5 - 118172923\,w^4 + 3790389\,w^3 \\
&+ 186393\,w^2 - 5138\,w + 931
\end{aligned}$$

$$
\begin{aligned}
P_3(w) =\ & 2811651371827200000\, w^{27} - 41266384607379456000\, w^{26} \\
& + 24773736838201344000\, w^{25} + 11079955148990054400\, w^{24} \\
& - 4824757963497799680\, w^{23} - 848931834754990080\, w^{22} \\
& - 1254549778859409408\, w^{21} + 332436165839831040\, w^{20} \\
& + 230844285233797120\, w^{19} + 23397830675772928\, w^{18} \\
& - 35731466794253184\, w^{17} - 7015978530910848\, w^{16} \\
& + 3334508426832544\, w^{15} + 129769752723552\, w^{14} \\
& + 64823043832572\, w^{13} + 9687172326860\, w^{12} - 27269045355036\, w^{11} \\
& + 668812543530\, w^{10} + 1232145138001\, w^{9} + 217652915670\, w^{8} \\
& - 113666172030\, w^{7} + 2636314376\, w^{6} + 2843958768\, w^{5} \\
& - 343848072\, w^{4} + 7774186\, w^{3} + 533670\, w^{2} - 21133\, w + 2254 \\
P_2(w) =\ & 8434954115481600000\, w^{28} - 130054335243485184000\, w^{27} \\
& + 101906251242799104000\, w^{26} + 14913046820133273600\, w^{25} \\
& - 16311072130508390400\, w^{24} + 3172467314793676800\, w^{23} \\
& - 7523898588462022656\, w^{22} + 4342034847529598976\, w^{21} \\
& + 349567990475905024\, w^{20} - 1019978058716864000\, w^{19} \\
& + 135175191464703616\, w^{18} + 154325934113136384\, w^{17} \\
& - 39182557846936736\, w^{16} - 6064402640954016\, w^{15} \\
& - 721601939503388\, w^{14} + 778716878219200\, w^{13} + 414373001033428\, w^{12} \\
& - 145383247830248\, w^{11} - 7935890967120\, w^{10} + 857310545482\, w^{9} \\
& + 2441550352242\, w^{8} - 452151200437\, w^{7} - 32488744995\, w^{6} \\
& + 15597930449\, w^{5} - 1389870493\, w^{4} + 21276545\, w^{3} \\
& + 2254771\, w^{2} - 106015\, w + 7987 \\
P_1(w) =\ & 421747705774080000\, w^{28} - 6710038906798080000\, w^{27} \\
& + 4873843971588096000\, w^{26} + 1036215739573862400\, w^{25} \\
& - 616703179652628480\, w^{24} - 120979270258114560\, w^{23} \\
& + 364788984254619648\, w^{22} - 102347117319833952\, w^{21} \\
& - 201098747980344320\, w^{20} + 95380882997566464\, w^{19} \\
& - 24685797215423923 2\, w^{18} - 15239461402380854 4\, w^{17} \\
& + 55269675209779600\, w^{16} + 3979478375375692\, w^{15} \\
& + 612261498482876\, w^{14} - 759726033790536\, w^{13} - 527516229973280\, w^{12} \\
& + 188404041710426\, w^{11} + 5343977363342\, w^{10} - 599703686985\, w^{9} \\
& - 2917831385349\, w^{8} + 538227200312\, w^{7} + 39103736504\, w^{6} \\
& - 18721639514\, w^{5} + 1671294354\, w^{4} - 25882894\, w^{3} \\
& - 2676882\, w^{2} + 122171\, w - 9261 \\
P_0(w) =\ & -32133158535168000\, w^{26} + 695978420915404800\, w^{25} \\
& - 544662214410240000\, w^{24} + 3609286260943749120\, w^{23} \\
& - 7536908069936824320\, w^{22} + 10191253169267146752\, w^{21} \\
& + 2316960321752137728\, w^{20} - 8808777472942522368\, w^{19}
\end{aligned}
$$

$$+ 2190279912630165504\, w^{18} + 1472339197313387520\, w^{17}$$
$$- 535796399651893248\, w^{16} - 40698424759766784\, w^{15}$$
$$- 1761836408896896\, w^{14} + 6557432066372256\, w^{13}$$
$$+ 5053923219182592\, w^{12} - 1884372068029344\, w^{11} - 14699422810656\, w^{10}$$
$$+ 1982678836896\, w^{9} + 27382956586968\, w^{8} - 5031984639480\, w^{7}$$
$$- 368590752120\, w^{6} + 176119304760\, w^{5} - 15746069112\, w^{4}$$
$$+ 246121272\, w^{3} + 24962808\, w^{2} - 1111320\, w + 84672$$

9. Appendix B

In this Appendix, we derive the $Y^{(n)}$ expansion (19). At the first step, the $n-1$ dimensional integral is translated to a n dimensional one, the angular constraint (7) being taken into account by a Dirac delta distribution $\delta(\sum_{i=1}^{n} \phi_i)$ which is Fourier expanded as (with $Z_p = \exp(i\, \phi_p)$)

$$2\pi \cdot \delta(\sum_{i=1}^{n} \phi_i) \; = \; \sum_{k=-\infty}^{\infty} \Bigl(Z_1\, Z_2\, \cdots\, Z_n\Bigr)^k \qquad (91)$$

The integrals (14) read

$$Y^{(n)}(w) = \sum_{k=-\infty}^{\infty} \int_0^{2\pi} \frac{d\phi_1}{2\pi} \int_0^{2\pi} \frac{d\phi_2}{2\pi} \cdots \int_0^{2\pi} \frac{d\phi_n}{2\pi} \cdot \Bigl(\prod_{i=1}^{n} y_i^2 \cdot Z_i^k\Bigr)$$
$$= \sum_{k=-\infty}^{\infty} \Bigl(\int_0^{2\pi} \frac{d\phi}{2\pi}\, y^2 \cdot Z^k\Bigr)^n \qquad (92)$$

From the Fourier expansion (16), one has the trivial integration rule

$$\int_0^{2\pi} \frac{d\phi}{2\pi} \left(y^2\, Z^k\right) \; = \; 4\, w^2 \cdot D(k) \qquad (93)$$

One gets

$$Y^{(n)}(w) \; = \; (4w^2)^n \cdot \sum_{k=-\infty}^{\infty} D(k)^n \qquad (94)$$

and with the definition (17), we finally obtain (19).

10. Appendix C

In this Appendix we give for the y^2-product model, the differential equation for $n = 4$ and the polynomials R_1 and R_2 corresponding to the solution $Y^{(5)}(w)$. The differential equation for $Y^{(4)}(w)$ reads

$$a_2\, \frac{d^2}{dw^2} Y^{(4)} \, + a_1\, \frac{d}{dw} Y^{(4)} \, + a_0\, Y^{(4)} \; = \; 0 \qquad (95)$$

where

$$a_2 \; = \; \left(1 - 16\, w^2\right)^2 \left(1 - 4\, w^2\right) \left(1 - 20\, w^2 + 16\, w^4 - 16\, w^6\right)\; w^2 \cdot P_2(w),$$
$$a_1 \; = \; 2\, w \left(1 - 16\, w^2\right)\; P_1(w), \qquad a_0 = 8\, P_0(w)$$

with:

$$
\begin{aligned}
P_2 \, =& \, -15 + 1086\, w^2 - 28106\, w^4 + 328716\, w^6 - 1946216\, w^8 \\
&+ 4791440\, w^{10} - 4697088\, w^{12} + 8682368\, w^{14} - 17308416\, w^{16} \\
&+ 1781760\, w^{18}, \\
P_1 \, =& \, 75 - 7716\, w^2 + 319862\, w^4 - 7044848\, w^6 + 92471304\, w^8 \\
&- 761556288\, w^{10} + 3861020800\, w^{12} - 11558422912\, w^{14} \\
&+ 22069511424\, w^{16} - 42440728576\, w^{18} + 101426089984\, w^{20} \\
&- 147848548352\, w^{22} + 72672886784\, w^{24} - 17528389632\, w^{26} \\
&+ 2736783360\, w^{28}, \\
P_0 \, =& \, -45 + 5400\, w^2 - 261002\, w^4 + 6787240\, w^6 - 107032250\, w^8 \\
&+ 1079592480\, w^{10} - 6995904504\, w^{12} + 28478625024\, w^{14} \\
&- 72563446848\, w^{16} + 142196898048\, w^{18} - 402528466944\, w^{20} \\
&+ 1067417646080\, w^{22} - 1296486492160\, w^{24} + 418540593152\, w^{26} \\
&- 55343579136\, w^{28} + 13683916800\, w^{30}.
\end{aligned}
$$

For the solution $Y^{(5)}(w)$, the polynomial $P_1^{(5)}$ and $P_2^{(5)}$ are given in (23) and

$$
\begin{aligned}
R_1^{(5)} \, =& \, 15 - 950\, w^2 - 550\, w^3 + 25581\, w^4 + 14856\, w^5 - 363086\, w^6 \\
&- 57174\, w^7 + 2727087\, w^8 - 1243960\, w^9 - 9109540\, w^{10} \\
&+ 10083482\, w^{11} + 8271200\, w^{12} - 23345456\, w^{13} - 2557976\, w^{14} \\
&+ 19604672\, w^{15} + 3295040\, w^{16} - 1531392\, w^{17} - 513024\, w^{18}, \\
R_2^{(5)} \, =& \, 3 + 6\, w - 103\, w^2 - 292\, w^3 + 1207\, w^4 + 2214\, w^5 - 7360\, w^6 \\
&+ 3674\, w^7 + 6896\, w^8 - 21328\, w^9 - 13080\, w^{10} - 96\, w^{11} - 256\, w^{12}.
\end{aligned}
$$

11. Appendix D

The differential equation satisfied by $\Phi_D^{(3)}$ reads:

$$
\sum_{n=1}^{4} a_n(w) \cdot \frac{d^n}{dw^n} F(w) \;\; = \;\; 0 \qquad (96)
$$

where

$$
\begin{aligned}
&a_4(w) = \, (1-w)\,(1+2w)\,(1-4w)\,(1+4w)\,\left(1+3w+4w^2\right)\, w^2 \cdot P_4(w), \\
&a_3(w) = w \cdot P_3(w), \qquad a_2(w) = P_2(w), \qquad a_1(w) = P_1(w)
\end{aligned}
$$

with

$$
\begin{aligned}
P_4(w) \, =& \, -18 - 126\, w - 536\, w^2 - 581\, w^3 + 11332\, w^4 + 56216\, w^5 \\
&+ 141103\, w^6 + 316146\, w^7 + 324516\, w^8 - 102512\, w^9 + 512\, w^{10} \\
&+ 104448\, w^{11}, \\
P_3(w) =& \, -36 - 450\, w - 270\, w^2 + 17199\, w^3 + 110892\, w^4 + 331122\, w^5 \\
&- 365893\, w^6 - 9173304\, w^7 - 40917443\, w^8 - 92069955\, w^9 \\
&- 128675122\, w^{10} - 89628548\, w^{11} + 223226064\, w^{12} \\
&+ 725436224\, w^{13} + 509586688\, w^{14} - 185729024\, w^{15} \\
&+ 25788416\, w^{16} + 133693440\, w^{17},
\end{aligned}
$$

$$
\begin{aligned}
P_2(w) \;=\;& 36 + 396\, w + 6114\, w^2 + 59550\, w^3 + 302337\, w^4 + 857367\, w^5 \\
& + 887406\, w^6 - 6754260\, w^7 - 39314109\, w^8 - 48807357\, w^9 \\
& + 175442484\, w^{10} + 683394108\, w^{11} + 1327431600\, w^{12} \\
& + 2198750784\, w^{13} + 1607543040\, w^{14} - 479219712\, w^{15} \\
& + 35880960\, w^{16} + 320864256\, w^{17}, \\
P_1(w) \;=\;& 108 + 1512\, w + 24282\, w^2 + 230094\, w^3 + 1072500\, w^4 \\
& + 3034020\, w^5 + 6713730\, w^6 + 7870578\, w^7 + 26737134\, w^8 \\
& + 253594938\, w^9 + 799267644\, w^{10} + 1253301264\, w^{11} \\
& + 1490308224\, w^{12} + 1015581696\, w^{13} - 338632704\, w^{14} \\
& - 1179648\, w^{15} + 160432128\, w^{16}.
\end{aligned}
$$

The differential equation satisfied by $\Phi_D^{(4)}$ reads:

$$
\sum_{n=1}^{4} a_n(w) \cdot \frac{d^n}{dw^n} F(w) \;=\; 0 \tag{97}
$$

where

$$
\begin{aligned}
a_4(w) \;=\;& w^3\,(1+4\,w)\,(1+2\,w)\,(1-2\,w)\,(1-4\,w) \cdot P_4(w), \\
a_3(w) \;=\;& 2\,w^4 \cdot P_3(w), \quad a_2(w) = 3w \cdot P_2(w), \quad a_1(w) = 3 \cdot P_1(w)
\end{aligned}
$$

with

$$
\begin{aligned}
P_4(w) \;=\;& 1 + 19\, w^2 + 314\, w^4 + 512\, w^6, \\
P_3(w) \;=\;& -81 - 1010\, w^2 - 1756\, w^4 + 89408\, w^6 + 114688\, w^8, \\
P_2(w) \;=\;& -1 - 53\, w^2 + 98304\, w^{10} + 117888\, w^8 + 742\, w^4 + 24896\, w^6, \\
P_1(w) \;=\;& 1 + 53\, w^2 + 2042\, w^4 + 15616\, w^6.
\end{aligned}
$$

12. Appendix E

In this Appendix, we give the proof for the solutions of the Landau conditions. One may use the form of the integrand given in (56), we instead use another form to show clearly how the singularities of the "Galois" companions are reproduced.

The variable x_j given in (6) from the definition (52) can be written as:

$$
x_j(\phi_j) \;=\; \exp\,(i\,\zeta_j) \tag{98}
$$

The integral (36) then becomes

$$
\Phi_D^{(n)} \;=\; \frac{1}{n!} \int_0^{2\pi} \frac{d\phi}{2\pi}\, \frac{i}{\tan(\psi/2)}, \qquad \psi = \;(n-1)\zeta\, + \zeta_n \tag{99}
$$

for which, the Landau conditions read:

$$
\begin{aligned}
& \alpha_1 \cdot \psi \;=\; 0, \qquad\qquad \alpha_2 \cdot \phi = \; 0, \\
& \alpha_3 \cdot \sin\,(\zeta)^2 \;=\; 0, \qquad\qquad \alpha_4 \cdot \sin\,(\zeta_n)^2 \;=\; 0, \\
& \alpha_1 \cdot \psi^{'} \;+\; \alpha_2 \;+\; 2\alpha_3 \cdot \sin(\phi)\cos(\zeta) \\
& \qquad\qquad +\; 2\alpha_4 \cdot \sin((n-1)\phi)\cos(\zeta_n) \;=\; 0
\end{aligned} \tag{100}
$$

where the parameters α_i, $i = 1, \cdots 4$, should not be, all, equal to zero. In the sequel, we give the proofs along the possible values (zero or not) of the two parameters α_3 and α_4.

12.1. $\alpha_3 = \alpha_4 = 0$

In this configuration, we are discarding the conditions on the branch points. The Landau conditions for α_1 and α_2 not both equal to zero give:

$$\zeta_n + (n-1)\zeta \; = \; 0, \qquad\qquad \phi \; = \; 0 \tag{101}$$

or:

$$\zeta_n + (n-1)\zeta \; = \; 0, \qquad\qquad {\frac{\sin((n-1)\phi)}{\sin(\zeta_n)}} \; = \; {\frac{\sin(\phi)}{\sin(\zeta)}} \tag{102}$$

These equations correspond (with $\sin(\zeta) \neq 0$), respectively, to an *end-point* singularity and to a *pinch* singularity. The condition (101) being contained in (102) for $\phi = 0$, one may consider the last.

The condition $\zeta_n + (n-1)\zeta = 0$ is *equivalent* to

$$\cos(\zeta_n) \; = \; \cos\left((n-1)\zeta\right), \tag{103}$$

$$\sin(\zeta_n) \; = \; -\sin\left((n-1)\zeta\right). \tag{104}$$

Using (52) for $\cos(\zeta_n)$ written as

$$\cos(\zeta_n) \; = \; {\frac{1}{2w}} - \cos((n-1)\phi) \; = \; {\frac{1}{2w}} - T_{n-1}\left(\cos(\phi)\right) \tag{105}$$

one obtains for (103):

$$\frac{1}{2w} - T_{n-1}(\cos(\phi)) \; = \; \cos\Big((n-1)\cos^{-1}\left(\frac{1}{2w} - \cos(\phi)\right)\Big) \tag{106}$$

Denoting $v = \cos(\phi)$, and by definition of the Chebyshev polynomials, one gets

$$\frac{1}{2w} - T_{n-1}(v) \; = \; T_{n-1}\Big(\frac{1}{2w} - v\Big) \tag{107}$$

which is the first condition given in (64). One has

$$\sin\left((n-1)\zeta\right) \; = \; \sqrt{1 - \left(\frac{1}{2w} - \cos(\phi)\right)^2} \cdot U_{n-2}\left(\frac{1}{2w} - \cos(\phi)\right)$$

and taking $\sin(\zeta_n)$ as

$$\sin(\zeta_n) \; = \; \sqrt{1 - \cos(\zeta_n)^2} = \sqrt{1 - \Big(\frac{1}{2w} - T_{n-1}\left(\cos(\phi)\right)\Big)^2} \tag{108}$$

the condition (104) gives

$$\sqrt{1 - \Big(\frac{1}{2w} - T_{n-1}(v)\Big)^2} + \sqrt{1 - \Big(\frac{1}{2w} - v\Big)^2} \cdot U_{n-2}\left(\frac{1}{2w} - v\right) \; = \; 0 \tag{109}$$

which is the third condition given in (64) and is a constraint to (107).

The second condition in (102) is automatically satisfied for $\phi = 0$ and $\phi = \pi$. Plugging these values in (107) and (109) gives respectively

$$1 \; + U_{n-2}\left(\frac{1}{2w} - 1\right) \; = \; 0, \tag{110}$$

$$\sqrt{1 - \left(\frac{1}{2w} - (-1)^{n-1}\right)^2} + \sqrt{1 - \left(\frac{1}{2w} + 1\right)^2} \cdot U_{n-2}\left(\frac{1}{2w} + 1\right) \; = \; 0$$

which are the constraints given in (62) and (63). The last have solutions only for n even.

At this point, the second condition in (102) has not been used. Using (104) in this last condition, one obtains

$$\frac{\sin((n-1)\left(\cos^{-1}(v)\right)}{\sin\left((n-1)\cos^{-1}(1/2w-z)\right)} + \frac{\sin\left(\cos^{-1}(v)\right)}{\sin\left(\cos^{-1}(1/2w-v)\right)} \; = \; 0 \qquad (111)$$

yielding:

$$U_{n-2}(v) \; - U_{n-2}\left(\frac{1}{2w} - v\right) \; = \; 0 \qquad (112)$$

which is the second of equation in (64).

Let us sum up for this case. Using the first condition in (102), giving *both* conditions (103) and (104), leads to a smaller set of singularities. While imposing the only condition (103) allows the appearance of the singularities of the "Galois companions".

12.2. $\alpha_3 \neq 0$

In this configuration, if $\alpha_4 = 0$, the condition $\sin(\zeta) = 0$ gives directly $w = \pm 1/4$.

For $\alpha_4 \neq 0$, we have $\sin(\zeta) = \sin(\zeta_n) = 0$, which, using the definition of $\cos(\zeta)$ and $\cos(\zeta_n)$ leads to

$$T_{n-1}\left(\frac{1}{2w} - (-1)^k\right) \; + (-1)^m \; - \frac{1}{2w} \; = \; 0, \qquad (113)$$

where k and m are integers.

12.3. $\alpha_3 = 0$ *and* $\alpha_4 \neq 0$

In this configuration, additional polynomials are obtained for $\alpha_1 \neq 0$ resulting in

$$(n-1)\zeta \; + \zeta_n \; = \; 0, \qquad \text{with,} \qquad \cos(\zeta)^2 \; = \; 1 \qquad (114)$$

leading to

$$T_{n-1}(v) \; - \left(\frac{1}{2w} \pm 1\right) \; = \; 0, \qquad \text{and} \qquad T_{n-1}\left(\frac{1}{2w} - v\right) \; \pm 1 \; = \; 0$$

which is solved by elimination of v.

13. Appendix F

The order one and order two differential operators occurring in the order-four linear differential operator

$$Ds \oplus \mathcal{L}_3(s) \; = \; Ds \oplus \Big(\mathcal{L}_2(s) \cdot \mathcal{L}_1(s)\Big) \qquad (115)$$

written in the variable s and corresponding to $\Phi_D^{(3)}$ read:

$$\mathcal{L}_1(s) \; = \; Ds \; - \frac{1}{2}\frac{Q_0}{Q_1}, \qquad \mathcal{L}_2(s) \; = \; Ds^2 \; + \frac{P_1}{P_2} \cdot Ds \; + \frac{P_0}{P_2}$$

with

$$
\begin{aligned}
Q_1 \,=\,\, & \big(1+s^2\big)\big(s^2+s+1\big) \\
& \times \big(2\,s^2-s+2\big)\big(2\,s^2+s+1\big)\big(s^2+s+2\big)\big(2\,s^2+s+2\big), \\
Q_0 \,=\,\, & \big(s^2-1\big) \\
& \times \big(8\,s^8+12\,s^7+26\,s^6+19\,s^5+32\,s^4+19\,s^3+26\,s^2+12\,s+8\big)
\end{aligned}
$$

and:

$$
\begin{aligned}
P_2 \,=\,\, & s\big(1+s^2\big)\big(2\,s^2+s+2\big)^2\big(s^2-1\big)^2\big(2\,s^2+s+1\big)\big(s^2+s+2\big), \\
& \big(s^2+s+1\big)\big(2\,s^2-s+2\big)\Big(64\,s^{16}+328\,s^{14}+596\,s^{13}+4434\,s^{12} \\
& +6621\,s^{11}+15377\,s^{10}+16897\,s^9+22822\,s^8+16897\,s^7+15377\,s^6 \\
& +6621\,s^5+4434\,s^4+596\,s^3+328\,s^2+64\Big), \\
P_1 \,=\,\, & \big(s^2-1\big)\big(2\,s^2+s+2\big)\big(1+s^2\big)\Big(2048\,s^{28}+3840\,s^{27}+17792\,s^{26} \\
& +63552\,s^{25}+381504\,s^{24}+1217888\,s^{23}+3351080\,s^{22}+6980836\,s^{21} \\
& +12444792\,s^{20}+18550693\,s^{19}+23711235\,s^{18}+26152837\,s^{17} \\
& +24357824\,s^{16}+19695118\,s^{15}+12653972\,s^{14}+7020790\,s^{13} \\
& +2315882\,s^{12}+789433\,s^{11}-161991\,s^{10}+312961\,s^9+197622\,s^8 \\
& +343348\,s^7+123248\,s^6+49472\,s^5-28512\,s^4-28608\,s^3 \\
& -17152\,s^2-3840\,s-1024\Big), \\
P_0 \,=\,\, & 2048\,s^{33}+3072\,s^{32}+17152\,s^{31}+69760\,s^{30}+526400\,s^{29} \\
& +1787232\,s^{28}+5094288\,s^{27}+11072344\,s^{26}+19781548\,s^{25} \\
& +27467354\,s^{24}+26981054\,s^{23}+5123635\,s^{22}-46141730\,s^{21} \\
& -134419668\,s^{20}-242023980\,s^{19}-354116927\,s^{18}-430640772\,s^{17} \\
& -463578366\,s^{16}-433931810\,s^{15}-371127411\,s^{14}-281456218\,s^{13} \\
& -203391640\,s^{12}-133972232\,s^{11}-89380041\,s^{10} \\
& -55343916\,s^9-34593136\,s^8-19033912\,s^7-9514544\,s^6-3824384\,s^5 \\
& -1169344\,s^4-213760\,s^3-768\,s^2+13824\,s+2048.
\end{aligned}
$$

The polynomials $D_0,\ D_K$ and D_E occurring in the solutions (82) of the order two linear differential operator $\mathcal{L}_2(s)$ are:

$$
\begin{aligned}
D_0 \,=\,\, & s\,(s-1)\,(1+s)\,\big(1+s^2\big)\big(2+s+2\,s^2\big)\big(2-s+2\,s^2\big) \\
& \big(1+s+s^2\big)\big(2+s+s^2\big)\big(1+s+2\,s^2\big), \\
D_K \,=\,\, & (s-1)\,(1+s)^2\big(1+s^2\big)^2 \\
& \big(8\,s^8+4\,s^7+22\,s^6+9\,s^5+93\,s^4+60\,s^3+100\,s^2+32\,s+32\big), \\
D_E \,=\,\, & 2\,\big(1+s^2\big)(1+s)^2\big(2+s+2\,s^2\big) \\
& \big(8\,s^6-2\,s^5+17\,s^4-10\,s^3+17\,s^2-2\,s+8\big).
\end{aligned}
$$

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